Econ 205A – Gorry Macro

Problem Set 1

The Neo-Classical Growth is Bounded

Show that there exists a number $\bar{k} > 0$ such that for any feasible allocation in the neoclassical growth model discussed in class that $k_t < \bar{k}$ for all t. Hint: combine the feasibility constant with capital accumulation equation and argue that conditions $c_t = 0$ and $h_t = 1$ imply maximal capital accumulation. Then use properties of F to show that there is a maximal amount of capital $\bar{k}$ for any initial capital stock k_0. A graph might help you.

Start with the first three equations in our model

(1) $y_t = F(k_t, h_t)$

(2) $c_t + i_t = y$

(3) $k_{t+1} = (1-\delta)k_t + y_t$

Setting $c_t = 0$ and $h_t = 1$ we can solve for k_{t+1} and get:

$$k_{t+1} = (1-\delta)k_t + f(k_t)$$

That is, "tomorrow's" capital is equal to:

"today's" capital decreased by depreciation

plus "today's" capital changed (think growth, increased etc) by our production function.

Our discrete time phase diagram looks like:

Describing the "Discrete Time Phase Diagram" above: With our Inada Conditions, we know that near t=0 the slope is nearly infinity, which is greater than one, the slope of our $k_{t+1} = k$ 45degree line. Also given our Inada conditions, as $t \to \infty$, k_t approaches zero. Meaning:

$$k_{t+1} = (1-\delta)k_t + f(k_t)$$

$$\frac{\delta k_{t+1}}{dk_t} = (1-\delta) + f'(k_t)$$

$$\lim_{k\to\infty} \frac{\delta k_{t+1}}{dk_t}(1-\delta) + (0)$$

So the slope is (1-delta), which—given our assumption $0 < \delta < 1$, is less than one.

That means there is a fixed point where the two lines intersect, where $k_{t+1} = (1-\delta)k_t + f(k_t)$. We'll call that $\hat{k}$

There are two scenarios. One where $k_0 < \hat{k}$ & one where $k_0 > \hat{k}$

$\boldsymbol{k_0 < \hat{k}}$ - When initial k is less than our k-hat, the slope of $(1-\delta)k_t + f(k_t)$ is greater than one, meaning k_{t+1} increases each period. As $t \to \infty, k_t \to \hat{k}$. Thus k is bounded below $\hat{k}$; any relevant level of initial capital will increase. (And I should note that by 'relevant', I mean for any level of capital greater than zero)

$\boldsymbol{k_0 > \hat{k}}$ - When initial k is greater than our k-hat, the slope of $(1-\delta)k_t + f(k_t)$ is less than 1 (Once again, assuming $0 < \delta < 1$; assuming depreciation exists). So when our initial capital is greater than k-hat, then k_{t+1} is less than one and capital will decrease. As $t \to \infty, k_t \to \hat{k}$. Thus k is bounded above $\hat{k}$: any level of capital where $k_0 > \hat{k}$ will decrease.

For five-year-olds; if there's a desert island and the only resource is coconuts, if the island is covered with coconuts—far more coconuts than the island can sustain—then the number of coconuts is sure to decrease—whether there is consumption or not. That means if your island continues on by itself, the dynamics of the island ensures there is max number of coconuts, it's initial number.

We are maximizing over a Bounded set.

Also of some note—and unrelated to the question—c is bounded as well. Because c is:

$$c_t = f(k) + (1-\delta)k_t - k_{t-1}$$

$\& \; we've \; shown \; our \; k \; values \; are \; bounded$

A continuous function of bounded functions is also bounded.

Discounting and Depreciation

For the growth model as discussed in class, determine the effects on increases in β and δ on the steady state capital stock. In each case, provide a brief explanation of the economic intuition behind the answer – what are the economic forces that underlie the result.

Our maximization problem's FOC was the following:

$$\beta^t u'(c_t)[f'(k_t) + (1-\delta)] - \beta^{t-1} u'\left(c_{(t-1)}\right) = 0$$

Dividing both sides by β^{t-1} and rearranging the variables a bit, we have:

$$u'(c_{t-1}) = \beta\, u'(c_t)[f'(k_t) + (1-\delta)]$$

Now, at our steady state $c^* = c_t = c_{t-1}$ and similarly, $k_t = k^*$ Thus:

$$u'(c^*) = \beta\, u'(c^*)[f'(k^*) + (1-\delta)]$$

Now, we don't really care about $u'(c^*)$, it's just a constant. And we want to find what $f'(k^*)$ equals in terms of β & δ. We should eliminate $u'(c^*)$ by dividing both sides by that constant. Thus:

$$1 = \beta[f'(k^*) + (1-\delta)]$$

Solving for $f'(k^*)$,

$$f'(k^*) = \frac{1}{\beta} - (1-\delta)$$

We can graph this,

Describing the above. Given that $0 < \beta \leq 1$, as β increases—as you care more about the future—$f'(k^*)$ *decreases*, and k* increases. If β decreases—meaning we discount the future more—$f'(k^*)$ *increases*, meaning our k* is smaller. For beta, I like to think of it as impatience, or short-sightedness. If you are willing to sacrifice today, if you truly value the future, then you'll value a higher steady state of capital (k*).

Mathematically:

$$f''(k^*)dk^* = -\frac{1}{\beta^2}d\beta \quad \rightarrow \quad \frac{dk^*}{d\beta} = -\frac{1}{f''(k^*)\beta^2}$$

because $f''(k^*) < 0$, *and* $\beta^2 > 0$, $\boldsymbol{\frac{dk^*}{d\beta}} > 0$, *a direct relationship*

δ –depreciation—is the reverse of β. As δ increases—and k decreases more in value in future—then you'll want to save more today. With $f'(k^*)$ *up*, k* will decrease. With depreciation less, k*, the sustainable level of capital—will be greater. For five-year-olds; if you have a farm of old fruit trees, aged, some sick, requiring special care to rally out a nice yield, (if you have high deprecation), then you can expect a lower steady state—a lower sustainable yield. Especially when compared to a farm of young fruit trees (a farm with low depreciation).

Mathematically:

$$f''(k^*)dk^* = d\delta \quad \rightarrow \quad \frac{dk^*}{d\delta} = \frac{1}{f''(k^*)},$$

because $f''(k^*) < 0$, $\boldsymbol{\frac{dk^*}{d\delta}} < 0$, *hense the inverse relationship*

Steady State with Cobb-Douglas ~~Utility~~

In the neoclassical growth model discussed in class, assume that the production function is given by $F(k,h) = Ak^{\theta}h^{1-\theta}$ where A > 0 and $0 \le \theta \le 1$. Describe expressions for the steady state level of the capital stock and consumption as function of A, $\theta, \beta, and\ \delta$.

Obviously there is one steady state at $k = 0$ & $c = 0$. But that isn't very interesting.

To solve this we are assuming h = 0. That turns $y_t = f(k_t, 1) \equiv f(k_t) = Ak^{\theta}$

To find something interesting, we need the first three equations of the NeoClassical Growth Model

(1) $y_t = F(k_t, 1) = Ak^{\theta}$

(2) $c_t + i_t = y_t$

(3) $k_{t+1} = (1-\delta)k_t + i_t$

At the steady state $k^* = k_{t-1} = k_t,\ h_t = h^*$ & $c^* = c_t$, and dropping out sub-t's.

$Working\ with\ (3): i = k^* - (1-\delta)k^*$

Plugging that into (2)

$$c = Ak^{*\theta} + (1-\delta)k^* - k^*$$

$$\max_{k} \sum_{t=0}^{\infty} \beta^t\, u\left(Ak^{*\theta} + (1-\delta)k^* - k^*\right)$$

$$k_0 = 0$$

First Order Condition generally (differentiating in terms of k) (from our notes):

$u'(c^*) = \beta\, u'(c^*) \cdot [f'(k^*) + (1-\delta)]$

Dividing all sides by $u'(c^*)$

$$1 = \beta\left[\theta Ak^{*\theta-1} + (1-\delta)\right] \rightarrow$$

$$\frac{1}{\beta} = \theta Ak^{*\theta-1} + (1-\delta)] \rightarrow$$

$$\theta Ak^{*\theta-1} = \frac{1}{\beta} - (1-\delta) \rightarrow$$

$$k^{*\theta-1} = \frac{\frac{1}{\beta} - (1-\delta)}{\theta A}$$

$$\boldsymbol{k^* = \left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{1}{\theta-1}} \rightarrow k^* = \left[\frac{\beta A\,\theta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\theta}}}$$

$$c^* = A\left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{\theta}{(\theta-1)}} + (1-\delta)\left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{1}{\theta-1}} - \left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{1}{\theta-1}}$$

$$\boldsymbol{c^* = A\left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{\theta}{(\theta-1)}} - \delta\left[\frac{\frac{1}{\beta} - (1-\delta)}{\theta A}\right]^{\frac{1}{\theta-1}}}$$

Habit Formation

In the neoclassical growth model discussed in class, instead of having utility given by:

$\sum_{t=0}^{\infty} \beta^t u(c_t)$, **assume that it is given by** $\sum_{t=0}^{\infty} \beta^t u(c_{t-1} c_t)$. **This additional term allows for costs associated with changing consumption over time. The general setup is called Habit formation. For this model, write down a problem that defines the set of efficient allocations. Derive a first order condition to characterize an efficient allocation. Finally, find a condition to characterize the steady state.**

I'm going to set-up a multi-period Lagrangian to tease out our first order conditions.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_{t-1} c_t) + \lambda[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}]$$

The Lagrangian over three periods:

$$\mathcal{L} = \beta^{t-1} u(c_{t-2}, c_{t-1}) + \lambda_{t-1}[f(k_{t-1}) + (1-\delta)k_{t-1} - c_{t-1} - k_t]$$
$$+\beta^t u(c_{t-1}, c_t) + \lambda_t[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}]$$
$$+\beta^{t+1} u(c_{t,} c_{t+1}) + \lambda_{t+1}[f(k_{t+1}) + (1-\delta)k_{t+1} - c_{t+1} - k_{t+2}]$$

First Order Conditions

$u'(c_{t-1}, c_t)_2$, the sub-2 means it's the partial for c_t, the second term, it's $\frac{\delta u}{\delta c_t} dc_t$

$$(1) \frac{\delta \mathcal{L}}{\delta c_t} = \beta^t u_2'(c_{t-1}, c_t) + \beta^{t+1} u_1'(c_t, c_{t+1}) - \lambda_t = 0$$

$$(2) \frac{\delta \mathcal{L}}{\delta k_{t+1}} = -\lambda_t + \lambda_{t+1}[f'(k_{t+1}) + (1-\delta)] = 0$$

$$(3) \frac{\delta \mathcal{L}}{\delta \lambda_t} = f(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0$$

Find the Condition that Characterizes the Steady State

$$(1) \lambda_t = \beta^t u_2'(c_{t-1}, c_t) + \beta^{t+1} u_1'(c_t, c_{t+1})$$
$$(2) \lambda_t = \lambda_{t+1}[f'(k_{t+1}) + (1-\delta)]$$

Combine (1) & (2)

$$\beta^t u_2'(c_{t-1}, c_t) + \beta^{t+1} u_1'(c_t, c_{t+1}) = \lambda_{t+1}[f'(k_{t+1}) + (1-\delta)]$$

From (1), $\lambda_{t+1} = \beta^{t+1} u_2'(c_t, c_{t+1}) + \beta^{t+2} u_1'(c_{t+1}, c_{t+2})$, now plugging that in,

$$\beta^t u_2'(c_{t-1}, c_t) + \beta^{t+1} u_1'(c_t, c_{t+1}) = [\beta^{t+1} u_2'(c_t, c_{t+1}) + \beta^{t+2} u_1'(c_{t+1}, c_{t+2})] * [f'(k_{t+1}) + (1-\delta)]$$

Pulling out β^t

$$u_2'(c_{t-1}, c_t) + \beta u_1'(c_t, c_{t+1}) = \beta u_2'(c_t, c_{t+1}) + \beta^2 u_1'(c_{t+1}, c_{t+2})[f'(k_{t+1}) + (1-\delta)]$$

Solving for the Steady State, and $u_1'(c_t, c_{t+1}) = u_1'(c_{t+1}, c_{t+2}) = u_1'(c^*, c^*)$ & $f'(k_{t+1}) = f'(k^*)$

$$\frac{u_2'(c^*, c^*) + \beta u_1'(c^*, c^*)}{\beta(u_2'(c^*, c^*) + \beta u_1'(c^*, c^*))} = f'(k^*) + (1-\delta)$$

$$\frac{1}{\beta} = f'(k^*) + (1-\delta)$$

Finally: $\boldsymbol{f'(k^*) = \frac{1}{\beta} - (1-\delta)}$ Just like in our non-habit formation problems.

k* is the same as we found for the neoclassical growth model example in class & question two. In the steady state consumers are equalizing the utility of consumption today and consumption tomorrow. Thus it does not matter if the model includes a vehicle for habit formation so long as it doesn't change aspects the discount rate or production function.

Also of some note though note specifically asked, although you do have the same SS, the dynamics of the path to that SS will be different when compared to the neoclassical growth model we learned in class. With habit formation—assuming k does not start right at k*—the path to the SS would take a bit longer than in our first model. The utility function inclines consumption levels to tend more toward the previous level. Thus people used to spending a lot will continue to--& people inclined to saving a lot will continue to. However our model deals with time on an infinite horizon, so the Steady State in H.F. will, eventually, be the same as in our initial model

Durable Consumption Goods

In this problem we are going to add a durable consumption good to the growth model discussed in class. Preferences are now given by:

$$\sum_{t=0}^{\infty} \beta^t \, u(c_t, d_t)$$

Where c_t is non-durable consumption and d_t is the stock of durable consumption goods in period t.

Technology is given by:

$$y_t = F(k_t, h_t)$$
$$c_t + i_t^k + i_t^d = y_t$$
$$k_{t+1} = (1-\delta^k)k_t + i_t^k$$
$$d_{t+1} - (1-\delta^d)d_t + i_t^d$$
$$i_t^k \geq 0, i_t^d \geq 0, d_t \geq 0, k_t \geq 0$$

Initial endowments are k_0, d_0. All other variables are easily interpretable from the model discussed in class.

For this model, write down a problem that defines the set of efficient allocation. Derive first order conditions to characterize efficient allocation. Finally, find conditions to characterize the steady state for this problem.

Setting up or Lagrangian:

$$\mathcal{L} = \max_{c_t, k_t, d_t} \sum_{t=0}^{\infty} \beta^t \, u(c_t, d_t) + \lambda_t[(1-\delta^k)k_t + (1-\delta^d)d_t + f(k_t) - c_t - k_{t+1} - d_{t+1}\,] +$$
$$\beta^{t+1}u(c_{t+1}, d_{t+1}) + \lambda[(1-\delta^k)k_{t+1} + (1-\delta^d)d_{t+1} + f(k_{t+1}) - c_{t+1} - k_{t+2} - d_{t+2}] + \cdots$$

Our First Order Conditions are as follows

$$(1)\,\frac{\delta\mathcal{L}}{\delta c_t} = \beta^t u_1'(c_t, d_t) - \lambda_t = 0 \qquad \rightarrow \lambda_t = \beta^t u_1'(c_t, d_t) \qquad \rightarrow \lambda_{t+1} = \beta^{t+1} u_1'(c_{t+1}, d_{t+1})$$

$$(2)\,\frac{\delta\mathcal{L}}{\delta k_{t+1}} = \lambda_t + \lambda_{t+1}[(1-\delta^k) + f'(k_{t+1})] = 0$$

$$(3)\,\frac{\delta\mathcal{L}}{\delta d_t} = \beta^t u_2'(c_t, d_t) + \lambda_t(1-\delta^d) - \lambda_{t-1} = 0$$

Plugging equation (1) into (2)

$$\beta^t u_1'(c_t, d_t) = \beta^{t+1} u_1'(c_{t+1}, d_{t+1})[(1-\delta^k) + f'(k_{t+1})]$$

Detrending the model, imposing the steady state and solving for f'(k*) .

$$u_1'(c^*, d^*) = \beta u_1'(c^*, d^*)[(1-\delta^k) + f'(k^*)] \quad \rightarrow \quad \frac{1}{\beta} = (1-\delta^k) + f'(k^*)$$

$$\boldsymbol{f'(k^*) = \frac{1}{\beta} - (1-\delta^k)}$$

If you solved for $f'(k^*)$ by combining equation (2) & (3) together, you'll get the following:

$$\lambda_{t+1}[(1-\delta^k) + f'(k_{t+1})] = \beta^{t+1} u_2'(c_{t+1}, d_{t+1}) + \lambda_{t+1}(1-\delta^d)$$

$$\lambda_{t+1}[(1-\delta^k) - (1-\delta^d) + f'(k_{t+1})] = \beta^{t+1} u_2'(c_{t+1}, d_{t+1})$$

$$\lambda_{t+1} = \frac{\beta^{t+1} u_2'(c_{t+1}, d_{t+1})}{[(1-\delta^k) - (1-\delta^d) + f'(k_{t+1})]}$$

Change the period to t=1

$$\lambda_t = \frac{\beta^t u_2'(c_t, d_t)}{[(1-\delta^k) - (1-\delta^d) + f'(k_t)]}$$

Bringing in equation (1)

$$\frac{\cancel{\beta^t} u_2'(c_t, d_t)}{[(1-\delta^k) - (1-\delta^d) + f'(k_t)]} = \cancel{\beta^t} u_1'(c_t, d_t)$$

Imposing the steady state and solving for f'(k*)

$$\boldsymbol{f'(k^*) = \frac{u_2'(c^*, d^*)}{u_1'(c^*, d^*)} + \delta^k - \delta^d}$$

You can also combine equations (1) and (3) to find an interesting relationship.

$$(3)\ \frac{\delta \mathcal{L}}{\delta d_{t+1}} = \beta^{t+1} u_2'(c_{t+1}, d_{t+1}) + \lambda_{t+1}(1-\delta^d) - \lambda_t = 0$$

$$\beta^{t+1} u_2'(c_{t+1}, d_{t+1}) + \beta^{t+1} u_1'(c_{t+1}, d_{t+1})(1-\delta^d) = \beta^t u_1'(c_t, d_t) \quad *$$

Imposing the steady state and detrending,

$$\beta u_2'(c^*, d^*) + \beta u_1'(c^*, d^*)(1-\delta^d) = u_1'(c^*, d^*)$$

$$\boldsymbol{u_2'(c^*, d^*) = u_1'(c^*, d^*)[\frac{1}{\beta} - (1-\delta^d)]}$$

What does this tell you? The marginal rate of substitution between the utility of durable goods and the utility of consumption is expressed by the ratio $\frac{1}{\beta} - (1-\delta^d)$.

Looking at the following equation (just taken from work* above before the SS was imposed):

$$u_1'(c_t, d_t) = \beta\left(u_2'(c_{t+1}, d_{t+1}) + u_1'(c_{t+1}, d_{t+1})(1-\delta^d)\right)$$

Meaning a change in the utility of consumption today is equivalent to the discounted utility of durable goods tomorrow, plus utility from consumption, less durable goods depreciation.

I also think an interesting point is that we could have a utility function in the from u(c,k), where k, capital, could stand in as a catch-all for durable goods. Where with the model above utility is in part a function of the stock of durable goods, in this hypothetical model utility will be, in part, a function of the capital stock. However our model above allows for a couple of extra points of analysis. We can depreciate durable goods at a different rate, δ^d vs δ^k. Also k is changed by the production function, and durable goods are not.

Kind of interesting.

Feasible Steady States

The problem asks you to work out the set of feasible steady states in the neoclassical growth model and show that the "best" feasible steady state is not the same as the efficient steady state.

6.1 Using the feasibility constraint & law of motion for capital:

$$f(k_t) = c_t + i_t$$
$$k_{t+1} = (1-\delta)k_t + i_t$$

Write down an equation for pairs of sustainable steady state values of (c_{ss}, k_{ss})

$f(k_{ss}) - c_{ss} = +i_t$

$k_{ss} - (1-\delta)k_{ss} = +i_t$

Thus, linking them together,
$k_{ss} - (1-\delta)k_{ss} = f(k_{ss}) - c_{ss} \rightarrow$

$$\boldsymbol{k_{ss} = \frac{f(k^*) - c_{ss}}{\delta}}$$
$$\boldsymbol{c_{ss} = f(k^*) - \delta k_{ss}}$$

6.2 Define $\hat{k}$ as the value of capital such that the entire amount of capital is used to maintain the capital stock $(f(\hat{k}) = \delta\hat{k})$. Plot the graph of the equation from part (1) with k on the horizontal axis and c on the vertical axis.

Graph $c_{ss} = f(k^*) - \delta k_{ss}$

Describing above, as $k \rightarrow 0$, we know that $f(k^*) \rightarrow \infty$ (Inida conditions). We also know that as $k \rightarrow \infty$ $f(k^*) \rightarrow 0$. Because δ & k_{ss} are both positive numbers, the term $-\delta k_{ss}$ will be negative. Thus our graph at one point will have slope of negative-delta (which is less than one), thus our function will also cross the k-axis at the origin and one other point.

6.3 Solve for the steady state value of k that maximizes consumption in each period, call this $\bar{k}_{ss}$.

$\bar{k}_{ss}$ - our highest point for c is where:

$$\frac{dc}{dk} = f'(\bar{k}_{ss}) - \delta = 0 \quad where \quad f'(\bar{k}_{ss}) = \delta$$

If you recall, before $f'(k^*) = \frac{1}{\beta} - (1-\delta)$

Interestingly in this scenario, the discount rate beta has no impact on the steady state level of capital k*. It's as if Beta were equal to one.

6.4 Recall that the efficient steady state from the growth model in class is given by $f'(k^*) = \frac{1}{\beta} - (1-\delta)$. Compare $\bar{k}_{ss}$ with the efficient steady state k* from the growth model. How does consumption differ in the two cases? Suppose a consumer started with $\bar{k}_{ss}$ units of capital, what would she like to do to maximize her utility? Why?

Is $\bar{k}_{ss}$ less or more than k*? Well think about the **slope** of k* and $\bar{\bar{k}}_{ss}$.

$$f'(k^*) = \frac{1}{\beta} - (1-\delta) > f'(\bar{k}_{ss}) = \delta$$

As you can see from our graph, that will put k* below $\bar{k}_{ss}$. With k* less than k-bar, then c* is less than $\overline{c_{ss}}$.

All of the various possible steady state levels are expressed in the graph above. The actual values of k*, and accordingly c* are just a function $f'(k^*) = \frac{1}{\beta} - (1-\delta)$, a function of one's impatience-preference and the rate of depreciation.

$\bar{k}_{ss}$ seems consistent with the "Golden Rule" savings rate—the rate which maximizes the steady state level of consumption. Where the Neoclassical Growth Model includes a utility function that discounts the utility of future consumption. Implicitly, the neoclassical model assumes people are impatient and they want more consumption today – our 'best' level at $\bar{k}_{ss}$ & $\bar{c}_{ss}$ do not do this. At our savings and consumption levels k-bar & c-bar consumers are clearly not discounting future consumption: people are clearly "doing under others—*intertemporaly*—are they would have others do unto them—*intertemporaly*."