24/30

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Problem 1

$\exists\ \bar{k} > 0$ that for any feasible allocation $k_t < \bar{k}\ \forall t$ where $k_t < \bar{k}\ \forall t$

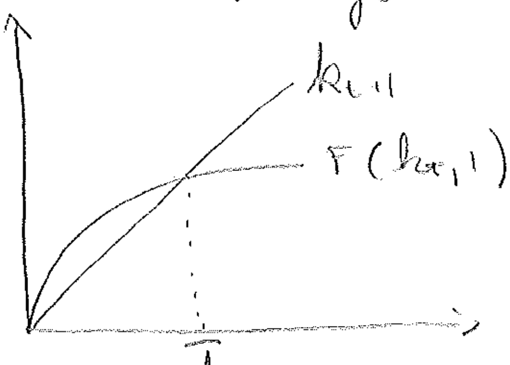
feasibility constraint $c_t + i_t = y_t$

$$k_{t+1} = (1-\delta)k_t + i_t$$

$c_t = 0 \quad h_t = 1$

↳ max cap accumulation

$i_t = y_t$



$\lim_{k\to 0} F_k(k,h) = \infty \qquad \lim_{k\to\infty} F_k(k,h) = (1-\delta) < 1$

$\exists\ \bar{k} > 0$ st $k_t < \bar{k}$

What is $\bar{k}$ for 3 cases

[2] Let $k = k^*$

then,

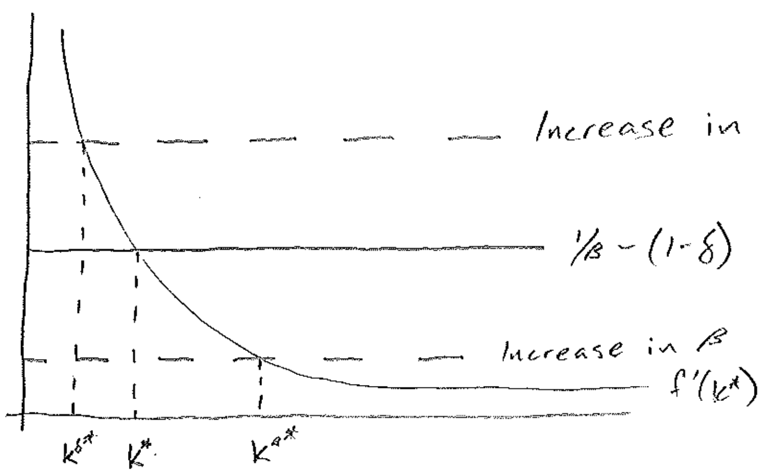
$$u'(c^*) = \beta u'(c^*)[f'(k^*) + (1-\delta)]$$

by the FOC. So,

$$1 = \beta[f'(k^*) + (1-\delta)]$$

$$\Longrightarrow f'(k^*) = 1/\beta - (1-\delta)$$

This shows that the steady state level of capital is dependent upon β and δ



This shows that as β increases, we see an increase in k^*. Economically, this makes sense; the more we value consumption in the future, the more we will save in the current time period.

An increase in δ induces a decrease in k^*. This also could have been predicted, it makes sense that we will consume more in the present time (thus lower capital in the future) if depreciation on this capital increases.

why

4

3) $F(k,h) = Ak^{\Theta}h^{1-\Theta}$, $A > 0$, $0 \leq \Theta \leq 1$, $h_t = 1$

$$\max \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) + (1-\delta)k_t - k_{t+1}\right)$$

$$\text{FOC}: \beta^t u'(c_t)\left[f'(k_t) + (1-\delta)\right] - \beta^{t-1} u'(c_{t-1}) = 0$$

$$\Rightarrow u'\left(f(h_{t-1}) + (1-\delta)k_{t-1} - k_t\right) = \beta u'\left(f(k_t) + (1-\delta)k_t - k_{t+1}\right) * \left[f'(k_t) + (1-\delta)\right]$$

SS: $k_t = k^*$, $c_t = c^*$,

Dividing by $u'(c^*)$:

$$1 = \beta\left(f'(k^*) + (1-\delta)\right) \Rightarrow f'(k^*) + (1-\delta)$$

$\Rightarrow F'(k^*) = \Theta A k^{*\Theta-1} h^{1-\Theta} = \Theta A k^{*\Theta-1}$ (since there is no leisure $\Rightarrow$ $h_t = 1$)

$$\Rightarrow \Theta A k^{*\Theta-1} = \frac{1}{\beta} - (1-\delta)$$

solving for k^*:

$$\Theta A k^{\Theta-1} = \frac{1-\beta(1-\delta)}{\beta}$$

$$k^{\Theta-1} = \frac{1-\beta(1-\delta)}{\Theta A \beta}$$

$$k^* = \left[\frac{1-\beta(1-\delta)}{\Theta A\beta}\right]^{\frac{1}{\Theta-1}} = \left[\frac{1/\beta-(1-\delta)}{\Theta A}\right]^{\frac{1}{\Theta-1}} \checkmark$$

$$c^* = f(k^*) + (1-\delta)k^* - k^* = f(k^*) - \delta(k^*)$$

$$c^* = f\left[\left(\frac{1/\beta-(1-\delta)}{\Theta A}\right)^{\frac{1}{\Theta-1}}\right] - \delta\left[\left(\frac{1/\beta-(1-\delta)}{\Theta A}\right)^{\frac{1}{\Theta-1}}\right]$$

$$c^* = A\left(\frac{1/\beta-(1-\delta)}{\Theta A}\right)^{\frac{\Theta}{\Theta-1}} - \delta\left[\left(\frac{1/\beta-(1-\delta)}{\Theta A}\right)^{\frac{1}{\Theta-1}}\right] \checkmark \quad S$$

Macro P. S. 1, #4

#4 - Problem defining set of efficient allocations

$$\max_{c_t, k_t, h_t, i_t, y_t} \sum_{t=0}^{\infty} \beta^t u(c_{t-1}, c_t)$$

S.t. eqn.'s (1) to (8)

(1) $y_t = F(k_t, h_t)$

(2) $c_t + i_t = y_t$

(3) $k_{t+1} = (1-\delta)k_t + i_t$

(4) $k_t \geq 0$

(5) $c_t \geq 0$

(6) $i_t \geq 0$

(7) $0 \leq h_t \leq 1$

(8) k_0 given

$\boxed{\text{- Derive F.O.C's}}$

$$\left.\begin{aligned} c_t &= f(k_t) + (1-\delta)k_t - k_{t+1} \\ c_{t-1} &= f(k_{t-1}) + (1-\delta)k_{t-1} - k_t \end{aligned}\right\} \text{From eqn.'s (1) to (8)}$$

$$\sum_{t=0}^{\infty} \beta^t u[c_{t-1}, c_t]$$

$$= \sum_{t=0}^{\infty} \beta^t u\left[f(k_{t-1}) + (1-\delta)k_{t-1} - k_t \; , \; f(k_t) + (1-\delta)k_t - k_{t+1}\right]$$

#4

#4
(cont.d)

→ k_t is in 3 time periods; $t-1$, t, $t+1$:

Write down Max problem you are solving

$$\beta^{t-1}u[(c_{t-2}, c_{t-1})] + [\beta^t u(c_{t-1}, c_t)] + [\beta^{t+1}(c_t, c_{t+1})]$$

F.O.C.

Note: u_1 is w.r.t 1st term,
u_2 is w.r.t 2nd term

$$\beta^{t-1}u_1(c_{t-2}, c_{t-1})0 + \beta^{t-1}u_2(c_{t-2}, c_{t-1})(-1) + \beta^t u_1(c_{t-1}, c_t)(-1) + \beta^t u_2(c_{t-1}, c_t)[f'(k_t) + (1-\delta)] + \beta^{t+1}u_1(c_t, c_{t+1}) \cdot [f'(k_t) + (1-\delta)] = 0$$

- Dividing by β^t

$$\frac{1}{\beta}u_2(c_{t-2}, c_{t-1})(-1) + u_1(c_{t-1}, c_t)(-1) + u_2(c_{t-1}, c_t)[f'(k_t) + (1-\delta)] + \beta u_1(c_t, c_{t+1}) \cdot [f'(k_t) + (1-\delta)] = 0$$

#4 Cont'd

At steady state:

$$u_2(c_{t-2}, c_{t-1}) = u_2(c_{t-1}, c_t) = \underline{\underline{u_2}} = \underline{\underline{u_2(c^*, c^*)}}$$

$$u_1(c_{t-2}, c_{t-1}) = u_1(c_t, c_{t+1}) = \underline{\underline{u_1}} = \underline{\underline{u_1(c^*, c^*)}}$$

Thus, F.O.C.:

$$-\frac{1}{\beta}u_2 - u_1 + u_2[f'(k_t) + (1-\delta)] + \boxed{\beta u_1[f'(k_t) + (1-\delta)]} = 0$$

$$u_2\left[(f'(k_t) + (1-\delta)) - \frac{1}{\beta}\right] + \beta u_1\left[f'(k_t) + (1-\delta) - \frac{1}{\beta}\right] =$$

$$\left[f'(k_t^*) + (1-\delta) - \frac{1}{\beta}\right] \cdot \left[u_2 + \beta u_1\right] = 0$$

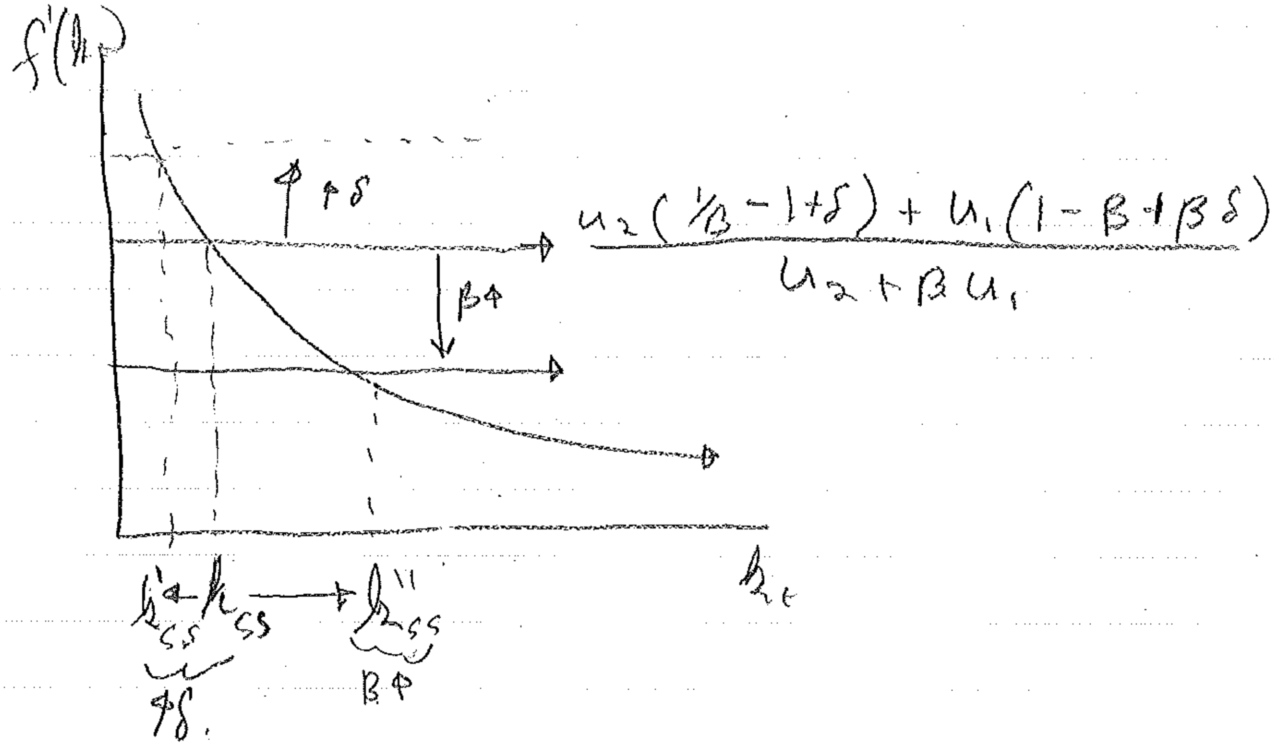
$$-\frac{1}{\beta}u_2 - u_1 + \underline{u_2 f'(k_t)} + u_2(1-\delta) + \underline{\beta u_1 f'(k_t)} + \beta u_1(1-\delta) = 0$$

$$f'(k_t)(u_2 + \beta u_1) = \frac{1}{\beta}u_2 + u_1 - u_2(1-\delta) - \beta u_1(1-\delta)$$

(1) $$f'(k_t) = \frac{\frac{1}{\beta}u_2 + u_1 - u_2(1-\delta) - \beta u_1(1-\delta)}{u_2 + \beta u_1}$$

(2) $$f'(k_t) = \frac{\frac{1}{\beta}u_2 + u_1 - u_2 + \delta u_2 - \beta u_1 + \beta u_1 \delta}{u_2 + \beta u_1}$$

$$f'(k_t^*) = \frac{u_2(\frac{1}{\beta}-1+\delta)+u_1(1-\beta+\beta\delta)}{u_2+\beta u_1} \qquad (1)$$



From the graph above, we see that as $\beta\uparrow$'s, k_{ss} shifts out to k''_{ss}. A similar process causes k_{ss} to decrease when β decreases. When δ increases, k_{ss} shifts left to k'_{ss}. This process is ecomomically interpreted as in #2.

Noting that u_1 is associated with the habit formation term, c_{t-1}, when $u_1 = 0$ (i.e. change in previous consumption does not effect utility in current period.), we get same result as in #2.

$$f'(k_t^*) = \frac{u_2(1/\beta - 1 + \delta) + 0(1 - \beta - \beta\delta)}{u_2 + \beta(0)}$$

$$\boxed{= 1/\beta - 1 + \delta}$$

Similarly, the term associated with u_2, c_t, we see that $u_2 = 0$ gives us the same result. $u_2 = 0$ says current period of consumption is predetermined by previous period

$$f'(k_{ss}) = \frac{0(1/\beta - 1 + \delta) + u_1(1 - \beta - \beta\delta)}{0 + \beta u_1}$$

$$\boxed{= 1/\beta - 1 + \delta}$$ ✓

Both of the above results formulate the same economic interpretation as in #2.

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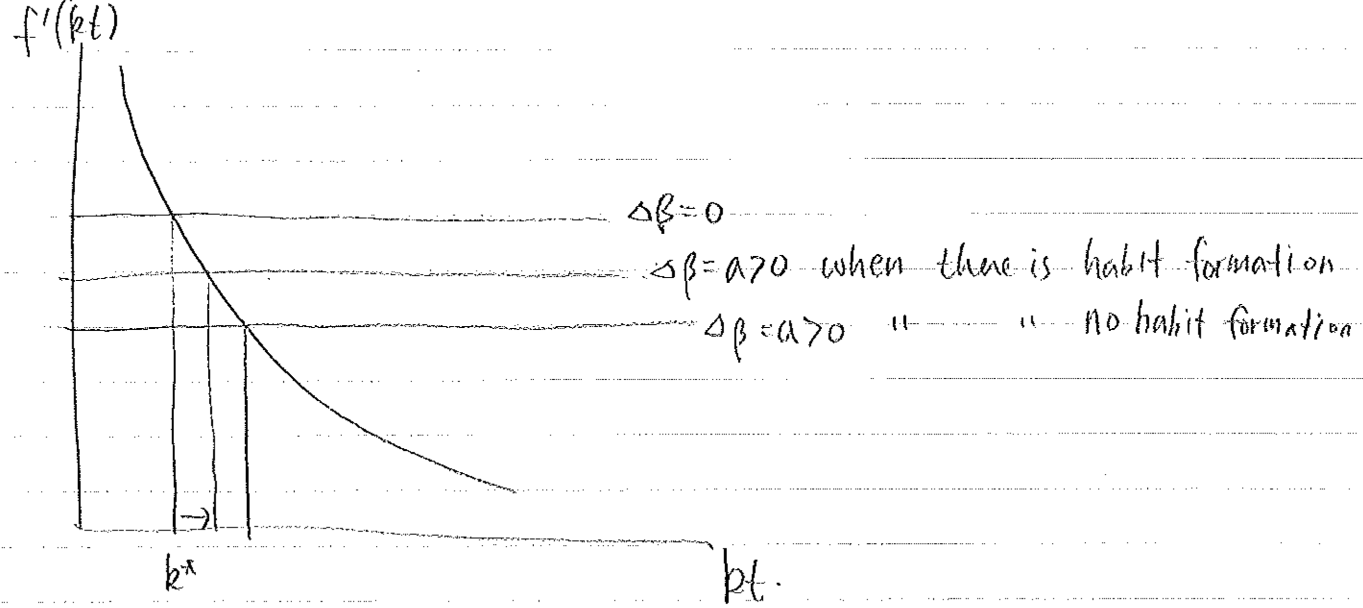
$$f'(k_t) = \frac{\frac{1}{\beta}u_2 + u_1}{u_2 + \beta u_1} - (1-\delta) \quad \text{(habit formation)}$$

$$f'(k_t) = \frac{1}{\beta} - (1-\delta) \quad \text{(when there is no habit formation).}$$



when β increases by $\Delta\beta = a > 0$, $\frac{\frac{1}{\beta}u_2 + u_1}{u_2 + \beta u_1}$ decreases less then $\frac{1}{\beta}$.

So, k^* increases less when there is habit formation.
This is because we care more previous consumption, so we save less with positive discounting rate shock.

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$c_t = y_t - i_t^k - i_t^d$

prob 5.

$$C_t = f(k_t) - i_t^k - i_t^d$$

$$= f(k_t) - (k_{t+1} - (1-\delta^k)k_t) - (d_{t+1} - (1-\delta^d)d_t)$$

assume $\bar{h} = 1$

$$\max_{k_t, d_t} \beta^t u(c_t, d_t)$$

foc wrt k_t

$u_1 = \frac{\partial u}{\partial c_t}$

$$\Rightarrow \beta^t \cdot u_1 [f'(k_t) + (1-\delta^k)] - \beta^{t-1} u_1 = 0$$

SS?

foc wrt d_t

$$\Rightarrow \beta^t \cdot u_1 [(1-\delta^d)] - \beta^{t-1} \cdot u_1 + \beta^t u_2 = 0$$

$u_2 = \frac{\partial u}{\partial d_t}$

$$\beta u_1 (1-\delta^d) - u_1 + \beta u_2 = 0$$

$$u_1(\beta - \beta\delta^d - 1) + \beta u_2 = 0.$$

$$\boxed{u_2 = \left(\frac{1}{\beta} + \delta^d - 1\right) \cdot u_1}$$

✓ $\delta^d \uparrow \rightarrow d_t \downarrow, c_t \uparrow$

$\beta \uparrow \rightarrow c_t \downarrow, d_t \uparrow$

Problem 5 continued

foc condition is

$$u_2 = \left(\frac{1}{\beta} + \delta^d - 1\right) \cdot u_1$$

If δ^d increases, $u_2 \uparrow$ or $u_1 \downarrow$ to maintain the equality so consumption of durable goods decrease or consumption of non-durable goods increase. In the economic sense, if depreciation rate increases, we consume less durable goods in the future, so we had better consume more non-durable goods or less durable goods now.

3

Problem 6

(6.1) Equation for pairs of sustainable steady state values of (c_{ss}, K_{ss})

$$f(k_t) = c_t + k_{t+1} - (1-\delta)k_t$$

$$f(k^*) = c_t^* + k_{t+1}^* - (1-\delta)k^*$$

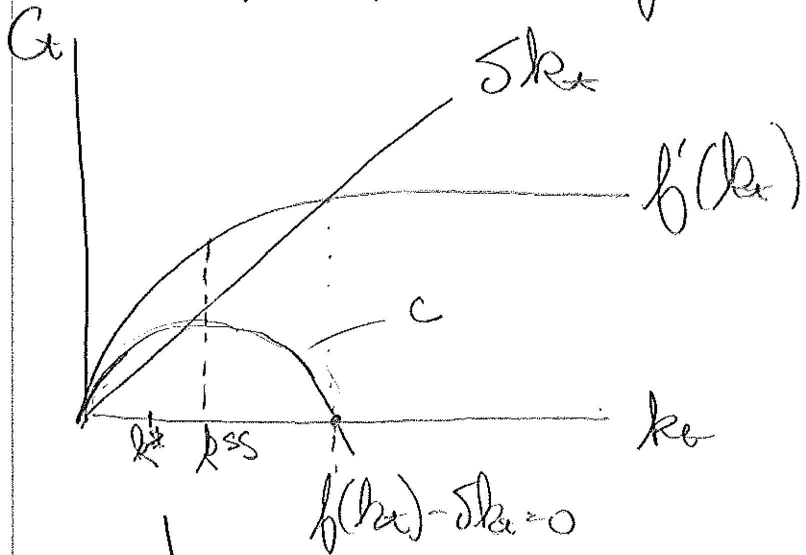
$$f(k^*) = c^* + \delta k^*$$

$$\rightarrow f(k^{ss}) = c^{ss} + \delta k^{ss}$$

$$\rightarrow c^{ss} = f(k^{ss}) - \delta k^{ss}$$

(6.2)

$$c = f(k_t) - i_t = f(k_t) - \delta k_t$$



$\downarrow$

$K^{ss} > K^*$ (for 6.4)

6.3

$$\max_{k_t} f(k_t) - \delta k_t$$

F.O.C. $f'(k_t) = \delta$

→ we are not discounting

6.4

$$f'(k^*) = \frac{1}{B} - (1-\delta) = \delta + \frac{1}{B} - 1 > \delta$$

$$f'(k^{ss}) = \delta$$

$$f'(k^*) > f'(k^{ss})$$

$k^{ss} > k^*$ (see graph on previous page)

We are consuming more w/k^{ss} b/c there is no discounting, thus the same c in the future is not worth less & we do not need to add 'interest' to give it same value in the k^{ss} scenario

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