Problem Set No 3

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(1)

1) An Arrow Debreu equilibrium would be a list of sequences

$\{y_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{p_t\}_{t=0}^{\infty}, \{h_t^f\}_{t=0}^{\infty}, \{k_t^f\}_{t=0}^{\infty}$

$\{c_{tj}\}_{t\geq 0}^{\infty}\ 0<j\leq n$ $\{i_{tj}\}_{t=0}^{\infty}\ 0<j\leq n$ $\{h_{tj}^c\}_{t=0}^{\infty}\ 0<j\leq n$ $\{k_{tj}^c\}_{t=0}^{\infty}\ 0<j\leq n$

Such that each consumer maximizes utility given prices p_t, r_t, w_t by choosing optimal levels of h_t^c, k_t^c, i_t, c_t

$c_t = f(k_t, 1) - i_t$

$k_{t+1}^c = (1-\delta)k_t + i_t$

$$\text{i.e. } \max \sum_{t=0}^{\infty} \beta^t u(c_{tj}) \quad \forall j \quad 0<j\leq n$$

Such that each consumer equals income with expenditures over time

$$\text{i.e. } \sum_{t=0}^{\infty} p_t c_{tj} + p_t i_{tj} = \sum_{t=0}^{\infty} r_t k_{tj}^c + w_t h_{tj}^c \quad 0<j\leq n$$

such that $c_{tj} \geq 0 \quad 0 \leq h_{tj} \leq 1 \quad k_{0j}$ given

$k_{t+1j} = (1-\delta)k_{tj} + i_{tj}$ and $\lim_{t\to\infty} c_{tj} = 0 \quad \forall j$

The firms problem remains the same, that is given prices p_t, w_t, r_t the firm chooses k_t^f, h_t^f, y_t in order to maximize profits

$$\text{i.e. } \max \sum_{t=0}^{\infty} p_t y_t - w_t h_t^f - r_t k_t^f$$

$$\text{s.t. } y_t = F(k_t^f, h_t^f) \quad k_t^f \geq 0, \ h_t^f \geq 0$$

and finally we need factor markets to clear that is:

$$h_t^f = \sum_{j=1}^{n} h_{tj}^c \quad \forall t \qquad \cancel{\sum_{t=0}^{\infty}} y_{tj} \geq \sum_{t=0}^{\infty} (c_{tj} + i_{tj}) \quad \forall t$$

$$k_t^f = \sum_{j=1}^{n} k_{tj}^c \quad \forall t$$

2) consumers problem assume $h_{tj}^c = 1 \quad \forall t, j$

$$\mathcal{L} = \max \sum_{t=0}^{\infty} \beta^t u(c_{tj}) + \lambda\left(\sum_{t=0}^{\infty} (p_t c_{tj} + p_t i_{tj}) - (r_t k_{tj}^c + w)\right)$$

replace i_{tj} with $k_{jt+1} - (1-\delta)k_{jt}$

$$\mathcal{L} = \max \sum_{t=0}^{\infty} \beta^t u(c_{tj}) + \lambda\left(\sum_{t=0}^{\infty} \left(p_t c_{tj} + p_t(k_{jt+1} - (1-\delta)k_{jt})\right) - (r_t k_{tj}^c + w)\right)$$

First Order Conditions

1.) $\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_{tj}) = \lambda p_t \Rightarrow \frac{\beta^t u'(c_{tj})}{\beta^{t-1} u'(c_{t-1j})} = \frac{\lambda p_t}{\lambda p_{t-1}} \Rightarrow \frac{\beta u'(c_{tj})}{u'(c_{t-1j})} = \frac{p_t}{p_{t-1}}$

2.) $\frac{\partial \mathcal{L}}{\partial k_t} = \lambda(p_{t-1}) = \lambda(r_t + p_t(1-\delta)) \Rightarrow \frac{p_{t-1}}{p_t} = \frac{r_t + p_t(1-\delta)}{p_t} \Rightarrow \frac{p_{t-1}}{p_t} = \frac{r_t}{p_t} + \frac{p_t(1-\delta)}{p_t} = \frac{p_{t-1}}{p_t} = \frac{r_t}{p_t} + (1-\delta)$

Firms Problem

$\max_{h_t, k_t} \sum_{t=0}^{\infty} p_t y_t - w h_t^f - r k_t^f$ s.t. $y_t = F(k_t^f, h_t^f)$ substitute in constraint

$\max_{h_t, k_t} \sum^{\infty} p_t (F(k_t^f, h_t^f)) - w h_t^f - r k_t^f$

First Order Conditions

3.) $\frac{\partial}{\partial k_t} = p_t F_k - r_t = 0 \Rightarrow F_k = r_t/p_t$ from 2 $\frac{r_t}{p_t} = \frac{p_{t-1}}{p_t} - (1-\delta) = F_k + (1-\delta) = \frac{u'(c_{tj+1})}{\beta u'(c_{tj})} \Rightarrow \boxed{F_k = \frac{1}{\beta} - 1 - \delta}$

4.) $\frac{\partial}{\partial h_t} = p_t F_h - w_t = 0 \Rightarrow F_h = w/p_t$

for s.s.

$p_t)$ $\beta \frac{u'(c_j^*)}{u'(c_j^*)} = \frac{p_t}{p_{t-1}} = \frac{p_t}{p_{t-1}} = \frac{\beta}{1}$ $\boxed{p_t = (p_{t-1})\beta}$

$w_t)$ $F_h = \frac{w_t}{p_t} = \frac{w_{t-1}}{p_{t-1}} = \frac{p_{t-1}}{p_t} = \frac{1}{\beta} \Rightarrow \boxed{w_t = \beta(w_{t-1})}$

$r_t)$ $F_k = \frac{r_t}{p_t} = \frac{r_{t-1}}{p_{t-1}} = \frac{p_{t-1}}{p_t} = \frac{1}{\beta} = \boxed{r_t = \beta r_{t-1}}$

3 Y

Problem 2. N person economy – non-identical preferences

Preferences are:

$$\sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \qquad i=1,2,\dots,N, \quad k_{i0} \text{ is given}$$

1. An AD competitive equilibrium for this economy is a list of sequences $\{c_{it}\}_{t=0}^{\infty}, \{k_{it}^c\}_{t=0}^{\infty}, \{h_{it}^c\}_{t=0}^{\infty}, \{i_{it}\}_{t=0}^{\infty}, \{k_t^f\}_{t=0}^{\infty}, \{h_t^f\}_{t=0}^{\infty},$

$\{y_t\}_{t=0}^{\infty}, \{p_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}$ such that

– for all consumers i, the consumer maximizes utility taking prices $\{p_t\}, \{w_t\}, \{r_t\}$ as given:

$$\max_{c_{it}, i_{it}, h_{it}^c, k_{it}^c} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \quad \forall i$$

$$\text{s.t. } \sum_{t=0}^{\infty} p_t(c_{it}+i_{it}) \le \sum_{t=0}^{\infty} w_t h_{it}^c + r_t k_{it}^c \quad \forall i$$

$$k_{it+1} = (1-\delta)k_{it}^c + i_{it}$$

$$0 \le h_{it}^c \le 1, \quad c_{it} \ge 0 \quad \forall i, t$$

$$\lim_{t\to\infty} p_t = 0$$

– the firm maximizes profit taking prices $\{p_t\}, \{w_t\}, \{r_t\}$ as given:

$$\max_{y_t, k_t^f, h_t^f} \sum_{t=0}^{\infty} p_t y_t - r_t k_t^f - w_t h_t^f \qquad \text{s.t. } y_t = F(h_t^f, k_t^f)$$

$$k_t^f \ge 0, \; h_t^f \ge 0 \quad \forall t$$

– markets clear:

$$h_t^f = \sum_{i=1}^{N} h_{it}^c$$

$$k_t^f = \sum_{i=1}^{N} k_{it}^c$$

$$y_t = \sum_{i=1}^{N} (c_{it} + i_{it}) \quad \forall t$$

2. A steady state equilibrium for this economy is a sequence of $k_i^* = (k_1^*, k_2^*, \dots, k_n^*)$ such that each k_i^* is constant over time. Note that in general $k_i^* \ne k_j^*$ and $\forall i \; k_{it+1} = (1-\delta)k_{it} + i_{it}$ and we have sequences $\{y_t\}, \{c_{it}\}, \{k_{it}^c\}, \{h_{it}^c\}, \{h_t^f\}, \{k_t^f\}, \{p_t\}, \{w_t\}, \{r_t\}$

3. Solving for:

i) Consumer max problem
assuming $k_{it}^c = 1$ and noting that $i_{it} = k_{it+1} - (1-\delta)k_{it}$

$$\max \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \quad \text{s.t.} \sum_{t=0}^{\infty} p_t c_{it} + p_t(k_{it+1} - (1-\delta)k_{it}) = r_t k_{it}^c - w_t$$

$$= \sum_{t=0}^{\infty} p_t(c_{it} + k_{it+1}) = r_t k_{it}^c + w_t + p_t(1-\delta)k_{it}^c$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) + \lambda(r_t k_{it}^c + w_t + p_t(1-\delta)k_{it}^c - p_t(c_{it} + k_{it+1}))$$

FOCs:
$$\frac{\partial \mathcal{L}}{\partial c_{it}} = \beta^t u_i'(c_{it}) = \lambda p_t \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial k_{it}^c} = \lambda p_{t-1} = \lambda(r_t + p_t(1-\delta)) \quad (2)$$

from (1): $\frac{\lambda p_t}{\lambda p_{t-1}} = \frac{\beta^t u_i'(c_{it})}{\beta^{t-1} u_i'(c_{it-1})} \Leftrightarrow \frac{p_{t-1}}{p_t} = \frac{1}{\beta}\frac{u_i'(c_{it-1})}{u_i'(c_{it})}$

from (2): $p_{t-1} = r_t + p_t(1-\delta) \Rightarrow \frac{p_{t-1}}{p_t} = \frac{r_t}{p_t} + (1-\delta)$

ii) Firm max. problem

$$\max \sum_{t=0}^{\infty} (p_t y_{it} - r_t k_{it}^f - w_t h_{it}^f) = \sum_{t=0}^{\infty} (p_t F(k_t^f, h_t^f) - r_t k_{it}^f - w_t h_{it}^f)$$

FOCs:
$$\frac{\partial \pi}{\partial k_{it}^f} = p_t F_k = r_t \Rightarrow F_k = \frac{r_t}{p_t} \quad (3)$$

$$\frac{\partial \pi}{\partial h_t^f} = p_t F_h = w_t \Rightarrow F_h = \frac{w_t}{p_t} \quad (4)$$

$$\frac{p_{t-1}}{p_t} = \frac{1}{\beta}\frac{u_i'(c_{it-1})}{u_i'(c_{it})} = \frac{r_t}{p_t} + (1-\delta) = \frac{p_{t-1}}{p_t} = \frac{u_i'(c_{it-1})}{\beta u_i'(c_{it})} = F_k + (1-\delta)$$

at SS $u_i'(c_{it-1}) = u_i'(c_{it}) = u_i'(c^*)$:

$$\frac{1}{\beta} = F_k^* + (1-\delta) \Rightarrow F_k^* = \frac{1}{\beta} - (1-\delta)$$

Given the N-person economy and looking at the problem ~~of~~ as capital stock per worker the aggreg. capital stock satisfies:

$f'\left(\frac{K^*}{N}\right) = \frac{1}{\beta} - (1-\delta)$ that is capital stock per worker may vary in the SS but average across all workers remains the same as previous model.

4. Social planner's problem:

$$\max_{c_i, i_t, h_t, k_t} \sum_{i=1}^{N} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \qquad 0 < \alpha_i < 1, \ \sum_{i=1}^{N} \alpha_i = 1$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t(c_{it} + i_{it}) \leq \sum_{t=0}^{\infty} r_t k_{it}^c + w_t h_{it}^c \quad \forall i$$

$$k_{it+1} = (1-\delta) k_{it} + i_{it}$$

$$0 \leq h_{it}^c \leq 1, \quad c_{it} \geq 0, \quad k_0 \text{ is given}$$

FOCs:
$$\frac{\partial \mathcal{L}}{\partial c_{it}} = \alpha_i \beta^t u_i'(c_{it}) = \lambda p_t \quad \Rightarrow$$

$$\Rightarrow \frac{\lambda p_t}{\lambda p_{t-1}} = \frac{\alpha_i \beta^t u_i'(c_{it})}{\alpha_i \beta^{t-1} u_i'(c_{it-1})} \Rightarrow \frac{p_{t-1}}{p_t} = \beta^{-1} \frac{u_i'(c_{it-1})}{u_i'(c_{it})} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial k_{it}^c} = \lambda p_{t-1} = \lambda(r_t k_t + (1-\delta) p_t) \quad \Rightarrow$$

$$\Rightarrow \frac{p_{t-1}}{p_t} = \frac{r_t}{p_t} + (1-\delta) \qquad (2)$$

$$\frac{p_{t-1}}{p_t} = \frac{1}{\beta} \frac{u_i'(c_{it+1})}{u_i'(c_{it})} = F_k - (1-\delta) \quad \text{at the SS} \quad F_k'\left(\frac{K}{N}\right) = \frac{1}{\beta} - (1-\delta)$$

That is given the restriction on the planner's weight the per capita capital stock remains the same we may have varying levels of capital across workers depending on the worker's utility function and the actions of the social planner but the average across workers remains unchanged regardless of the values of the planner's weights.

<u>Prob 3</u> (i) Endowment Economy
(ii) Two Agents
(iii) Identical preferences, given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_{it}) \qquad \forall i = 1, 2$$

$$\omega_i = (\omega_{i1}, \omega_{i2}, \dots)$$

1. Define an AD equilibrium

An Arrow-Debreu CE is a list of sequences $\{c_{it}\}_{t=0}^{\infty}$ $i=1,2$, $\{p_t\}_{t=0}^{\infty}$, $\{w_{it}\}_{t=0}^{\infty}$, $i=1,2$ s.t.

(i) Consumer Max: Taking prices $\{p_t\}_{t=0}^{\infty}$ and endowments $\{w_{it}\}_{t=0}^{\infty}, i=1,2$ as given, $\{c_{it}\}_{t=0}^{\infty}, i=1,2$ satisfies

already s.t.

$$\text{Max} \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t. } \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t w_{it} \quad \forall i = 1, 2,$$

$$c_{it} \geq 0, \quad w_{it} \text{ given } \forall t, \forall i$$

(ii) Markets Clear: $\quad c_{1t} + c_{2t} = w_{1t} + w_{2t} \quad \forall t$

2. Define a SOM equilibrium.

Introduce a bond market to enable consumers to transfer endowments to each other by means of borrowing/lending in order to ~~cor~~ smooth their own ~~inter-temp~~ consumption over time. Let the bonds issued in time period t be given by b_t and their issue price by q_t.

Then, a sequence of markets equilibrium is a list of sequences $\{c_{it}\}_{t=0}^{\infty}$ $i=1,2$, $\{p_t\}_{t=0}^{\infty}$, $\{w_{it}\}_{t=0}^{\infty}$ $i=1,2$,

$\{b_t\}_{t=0}^{\infty}$, $\{q_t\}_{t=0}^{\infty}$ s.t.

(i) Consumer Max: Taking prices $\{p_t\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$ and endowments $\{w_{it}\}_{t=0}^{\infty}$ $i=1,2$, $\{c_{it}\}_{t=0}^{\infty}$ $i=1,2$ and $\{b_t\}_{t=0}^{\infty}$ solve

$$\text{Max. } \beta^t u(c_{it}) \cancel{+ \lambda \sum_{t=0}^{\infty} (p_t w_{it} + p_t b_{t-1} - p_t c_t)}$$

$$\text{s.t. } \cancel{\sum^{\infty}} (p_t c_{it} + \cancel{p_t}\, q_t b_t) = \cancel{\sum_{t=0}^{\infty}} (p_t w_{it} + p_t b_{t-1}), \forall t, \forall i = 1, 2$$

$$c_{it} \geq 0, \quad w_{it} \text{ given } \quad \forall t, \forall i$$

[Note that the budget constraint for $t=0$ reduces to $p_0 c_{i0} + q_0 b_0 = p_0 w_{i0}$ as $b_{t-1} = 0$ at $t=0$.]

(ii) Markets clear: (a) $c_{1t} + c_{2t} = w_{1t} + w_{2t} \quad \forall t$

$\cancel{(b) \sum_{t=0}^{\infty} b_t = 0}$

(b) $\cancel{\Sigma}\ b_{1t} + b_{2t} = 0 \quad \forall t$

3. $u(c_t) = \log(c_t)$

$\omega_1 = (2, 1, 2, 1, 2, 1 \ldots)$

$\omega_2 = (1, 2, 1, 2, 1, 2, \ldots)$

3(a) AD Equilibrium:

$$\mathcal{L} = \beta^t \log(c_{it}) + \lambda \left[\sum_{t=0}^{\infty} (p_t w_{it} - p_t c_{it}) \right]$$

FOC:

(i) $\dfrac{\partial \mathcal{L}}{\partial c_{it}} = \dfrac{\beta^t}{c_{it}} - \lambda p_t = 0 \quad \forall t \qquad - (A)$

(ii) $\dfrac{\partial \mathcal{L}}{\partial \lambda} = \sum_{t=0}^{\infty} (p_t w_{it} - p_t c_{it}) = 0$

$$\Rightarrow \quad \frac{\beta^{t-1}}{c_{it-1}} = \lambda p_{t-1} \qquad - (B)$$

Dividing A by B :

$$\frac{\beta c_{it-1}}{c_{it}} = \frac{\lambda p_t}{\lambda p_{t-1}} \qquad - (C)$$

$\because$ we have a series of prices here, we can normalize one of them to 1. Set $p_0 = 1$

$$\therefore (C) \Rightarrow \frac{\beta c_{i0}}{c_{i1}} = p_1$$

$$\text{or} \quad \beta c_{i0} = p_1 c_{i1} \qquad - \text{C.1.}$$

$\because$ agent j has an identical utility function, we have an analogous relationship :

$$\beta c_{j0} = p_1 c_{j1} \qquad - \text{C. 2}$$

$\because$ we must have market clearing in each time period, $\parallel^{ly}$

$$\left.\begin{aligned} c_{i0} + c_{j0} &= w_{i0} + w_{j0} = 3 \\ c_{i1} + c_{j1} &= w_{i1} + w_{j1} = 3. \end{aligned}\right] D$$

Combining C1., C.2 and D, we get –

$$\beta(c_{i0} + c_{j0}) = p_1 (c_{i1} + c_{j1})$$

$$\Rightarrow \quad 3\beta = 3p_1$$

$$\beta = p_1$$

$\therefore$ by extension we get $p_2 = \beta^2$, $p_3 = \beta^3$ ... .

$$\Rightarrow \quad \boxed{p_t = \beta^t}$$

We also know from eq^n (C):

$$\frac{\beta C_{i\,t-1}}{C_{it}} = \frac{p_t}{p_{t-1}} \qquad \forall\, t.$$

$$\Rightarrow \quad \frac{\beta C_{i\,t-1}}{C_{it}} = \frac{\beta^t}{\beta^{t-1}}$$

$$\frac{\beta C_{i\,t-1}}{C_{it}} = \beta$$

$$\therefore \quad \boxed{\frac{C_{i\,t-1}}{C_{it}} = 1}$$

$\Rightarrow$ Consumption is constant over time for each agent. Let this constant value of consumption for agents 1 and 2 be given by c_1 & c_2 respectively.

Solving for c_1 :

Agent 1's budget constraint implies $\Rightarrow$

$$\sum_{t=0}^{\infty} p_t c_{1t} = \sum_{t=0}^{\infty} p_t w_{1t}$$

$$\sum_{t=0}^{\infty} \beta^t c_1 = \sum_{t=0}^{\infty} \beta^t w_{1t}$$

$$c_1 \sum_{t=0}^{\infty} \beta^t = 2 + \beta + 2\beta^2 + \beta^3 + 2\beta^4 + \beta^5 + \ldots\ldots$$

$$\frac{c_1}{1-\beta} = 2(1+\beta^2+\beta^4+\beta^6+\ldots) + \beta(1+\beta^2+\beta^4+\ldots)$$

$$\frac{c_1}{1-\beta} = \frac{2+\beta}{1-\beta^2}$$

$$\boxed{c_1 = \frac{2+\beta}{1+\beta}}$$

Solving for c_2:

$$\sum_{t=0}^{\infty} p_t c_{2t} = \sum_{t=0}^{\infty} p_t w_{2t}$$

$$\sum_{t=0}^{\infty} \beta^t c_2 = \sum_{t=0}^{\infty} \beta^t w_{2t}$$

$$c_2 \sum_{t=0}^{\infty} \beta^t = 1 + 2\beta + \beta^2 + 2\beta^3 + \beta^4 + \dots$$

$$\frac{c_2}{1-\beta} = (1+\beta^2+\beta^4+\beta^6) + 2\beta(1+\beta^2+\beta^4+\dots)$$

$$\frac{c_2}{1-\beta} = \frac{(1+2\beta)}{1-\beta^2}$$

$$\boxed{c_2 = \frac{1+2\beta}{1+\beta}}$$

$\therefore$ the AD equilibrium is given by:

$$\boxed{p_t = \beta^t, \quad c_{1t} = c_1 = \frac{2+\beta}{1+\beta}, \quad c_{2t} = c_2 = \frac{1+2\beta}{1+\beta}}$$

Notice that $c_1 > c_2 \quad \because \quad \beta < 1$

3(b) SoM Equilibrium:

$$\mathcal{L} = \beta^t \log(c_{it}) + \cancel{\beta} \sum_{t=0}^{\infty} \lambda_{it}(p_t w_{it} + p_t b_{it-1} - p_t c_{it} - q_t b_{it})$$

FOC:

(i) $$\frac{\partial \mathcal{L}}{\partial c_{it}} = \frac{\beta^t}{c_{it}} - \lambda_{it} p_t = 0 \qquad \forall t, \forall i$$

(ii) $$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_{it} q_t + \lambda_{it+1} p_{t+1} = 0 \qquad \forall t, \forall i$$

(iii) $$\frac{\partial \mathcal{L}}{\partial \lambda_t} = p_t w_{it} + p_t b_{it-1} - p_t c_{it} - q_t b_{it} \qquad \forall t, \forall i$$

$\therefore$ for agent 1:
(Note that the FOCs for agent 2 will be identical $\because$ of identical utility function and identical form of budget constraint) —

$$\text{(i)} \quad \frac{\beta^t}{c_t} = \lambda_t p_t \quad \Rightarrow \quad \lambda_t = \frac{\beta^t}{p_t c_t}$$

$$\text{(ii)} \quad \lambda_{t+1} p_{t+1} = \lambda_t q_t$$

$$\text{(i)} \Rightarrow \quad \lambda_{t+1} = \frac{\beta^{t+1}}{p_{t+1} c_{t+1}} \quad - \text{(iii)}$$

Substituting (i) and (iii) into (ii) —

$$\frac{\beta^{t+1}}{c_{t+1}} = \cancel{\beta^t} \; \frac{\beta^t q_t}{p_t c_t}$$

$$\Rightarrow \beta p_t c_t = q_t c_{t+1} \quad - \text{(iv)}$$

$$\frac{\beta p_t}{q_t} = \frac{c_{t+1}}{c_t} \quad - \text{iv a}$$

$$\cancel{\text{(i)} \Rightarrow \frac{\beta c_{t-1}}{c_t} = \frac{\lambda_t p_t}{\lambda_{t-1} p_t} \quad - \text{(v)}}$$

~~Substituting (ii) into (v) —~~

$$\cancel{\frac{\beta c_{t-1}}{c_t} = \frac{\lambda_{t-1} q_{t-1}}{\lambda_{t-1} p_t}}$$

$\because$ we have seperate market clearing conditions for each period, we can set 1 price per period as 1.

$\therefore \quad p_t = 1 \quad \forall t$

$\therefore$ iv (a) $\Rightarrow \quad \beta c_t = q_t c_{t+1}$

$\because$ the above equation would hold for both agents in each time period, we must have:

$$\beta (c_{1t} + c_{2t}) = q_t (c_{1\,t+1} + c_{2\,t+1})$$

Market Clearing $\Rightarrow$

$$3\beta = 3 q_t$$

$$\Rightarrow \quad \boxed{q_t = \beta} \quad \forall t \quad - \text{(v)}$$

$$\text{(iv) a \& (v)} \Rightarrow \quad \boxed{c_t = c_{t+1} \quad \forall t} \quad \text{(vi)}$$

$\therefore$ both agents would have constant consumption over time, given by c_1 for agent 1 and c_2 for agent 2

By Arrow's theorem, we have an equivalence between the AD CE and the SOM equilibrium in the presence of a one period bond market.

$$\therefore \quad c_1 = \frac{2+\beta}{1+\beta} \qquad c_2 = \frac{1+2\beta}{1+\beta}$$

For agent 1:

$$b_0 = \frac{2-c_1}{\beta} = \frac{2-\left(\frac{2+\beta}{1+\beta}\right)}{\beta} = \frac{\cancel{2}+\cancel{2}\beta-\cancel{2}-\beta}{\beta(1+\cancel{\beta})} = \frac{1}{1+\beta}$$

$$b_1 = \frac{\beta+2-(1+\beta)c_1}{\beta^2} = \frac{\beta+2-\frac{(1+\beta)(2+\beta)}{1+\beta}}{\beta^2}$$

Consider agent 1:

We know from his budget constraint that:

$$p_0 c_0^1 + q_0 b_0^1 = p_0 w_0$$

Substituting in values of p, q and w:

$$c_0^1 + \beta b_0^1 = 2 \qquad - (1)$$

At $t=1$: $p_1 c_1^1 + q_1 b_1^1 = p_1 w_1 + p_1 b_0^1$

$$\text{ie} \quad c_1^1 + \beta b_1^1 = 1 + b_0 \qquad - (2)$$

Subtracting (2) from (1):

$$\beta(b_0^1 - b_1^1) = 1 - b_0 \qquad (\because c_0^1 = c_1^1 = c^1)$$

$$(1+\beta) b_0^1 = 1 + \beta b_1^1$$

$$b_0^1 = \frac{1 + \beta b_1^1}{1+\beta}$$

Similarly for agent 2:

$$c_0^2 + \beta b_0^2 = 1$$

$$c_1^1 + \beta b_1^2 = 2 + b_0$$

$$\Rightarrow \quad \beta(b_0^2 - b_1^2) = -1 - b_0^2$$

$$(1+\beta) b_0^2 = \beta b_1^2 - 1$$

$$b_0^2 = \frac{\beta b_1^2 - 1}{1+\beta}$$

∵ Agent 1 is smoothing consumption over time by means of the bond, we must have:

$$2 - q b_0 = 1 + b_1$$

$$\Rightarrow 2 - \beta\left(\frac{1+\beta b_1}{1+\beta}\right) = 1 + b_1$$

$$\Rightarrow b_1 = \frac{1}{1+\beta+\beta^2}$$

$$b_0 = \frac{1+\beta b_1}{1+\beta}$$

$$= \frac{1 + \dfrac{\beta}{1+\beta+\beta^2}}{1+\beta}$$

$$= \frac{1+2\beta+\beta^2}{(1+\beta+\beta^2)(1+\beta)} = \frac{(1+\beta)^{\cancel{2}}}{(\cancel{1+\beta})(1+\beta+\beta^2)} = \frac{1+\beta}{1+\beta+\beta^2}$$

~~However,~~

∴ when agent 1 has endowment 2, his lending is given by $\frac{1+\beta}{1+\beta+\beta^2}$ and when his endowment is 1, his borrowing is $\frac{1+\beta b_1}{1+\beta}$ [∵ it is borrowing, we should have $-\frac{1+\beta b_1}{1+\beta}$]

Agent 2 will have the corresponding negatives.

∵ of AD equivalence, c_1 and c_2 are given by:

$$c_1 = \frac{2+\beta}{1+\beta} \qquad c_2 = \frac{1+2\beta}{1+\beta}$$

4) Growth Model with $u(c) = c$, Social Planner's problem:

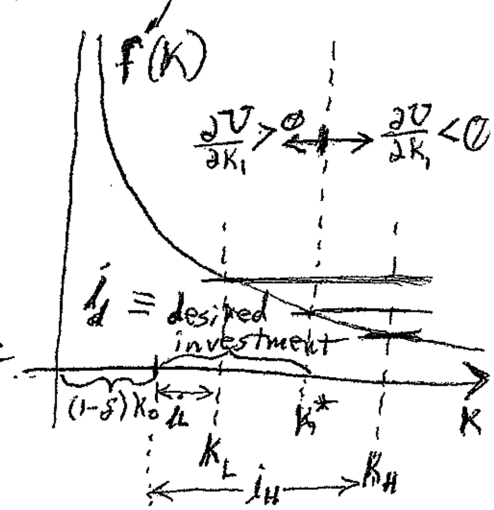
Show agent moves to s. state, K^* as quickly as possible.

$U \equiv \sum_{t=0}^{\infty} \beta^t u(c)$ e.g. $k_1 = k^*$ if $f(k_0) + (1-\delta)k_0 - K^* > 0$ otherwise $c_0 = 0$ & $k_2 = K^*$ if ...

$$\frac{\partial U}{\partial K_t} = \beta^t \underbrace{u'(c_t)}_{1}\left(f'(k_t) + (1-\delta)\right) - \beta^{t-1} \underbrace{u'(c_{t-1})}_{1} = \beta^t\left[f'(k_t) + (1-\delta) - \frac{1}{\beta}\right]$$

FOC; $f'(K_t^*) = \frac{1}{\beta} - (1-\delta)$ at the optimal utility.

& $K^* \equiv f'^{-1}\left(\frac{1}{\beta} - (1-\delta)\right)$



⇒ Planner maximizes U by setting $k_1 = K^*$.

Feasible for k_1? $(1-\delta)K_0$ automatically transfers to period 1 & $f(K_0)$ is available to invest.

Let $i_d \equiv$ desired investment $= K^* - (1-\delta)k_0$

A) If $f(k_0) + (1-\delta)k_0 - K^* \geq 0 \Rightarrow y_0 = f(k_0) \geq K^* - (1-\delta)k_0 = i_d$

Choose $c_0 \equiv y_0 - i_d$.

Note: 1) Consider choosing $c_0' > c_0 \Rightarrow i_L < i_d$ & $k_L \equiv (1-\delta)k_0 + i_L$

Low investment ⇒ low capital, k_L, ⇒ larger $f'(k_L)$

⇒ $\frac{\partial U}{\partial K_1} > 0$ ⇒ Planner could raise utility w/ larger investment!

2) Consider high investment, $i_H > i_d \Rightarrow \frac{\partial U}{\partial k_1} < 0 \Rightarrow$ better off consuming more!

B) If $f(k_0) < -(1-\delta)k_0 + K^*$ ⇒ Set $i_d = y_0$, $c_d = 0$

1) Likewise, choosing $i_L < i_d$ & $c_0 > 0 \Rightarrow \frac{\partial U}{\partial K_1} > 0$

⇒ Raising i_0 improves utility & the best you can do is w/ $c_0 = 0$.

2) In $t=1$ the planner now faces the identical problem & solves identically until $K_t = K^*$, ...

⑤ $u'(f(k)+(1-\delta)k-g(k)) = \beta u'(f(g(k))+(1-\delta)g(k)-g(g(k)))\,[f'(g(k))+(1-\delta)]$

1.) totally differentiate with respect to k:

$u''(c_{t-1})[f'(k)+(1-\delta)-g'(k)] = \beta u''(c_t)[f'(g(k))g'(k)+(1-\delta)g'(k)-g'(g(k))g'(k)]\cdot[f'(g(k))+(1-\delta)]$
$+\beta u'(c_t)[f''(g(k))\,g'(k)]-1$

2.) assume steady state i.e. $k=k^*$ and $k=g(k)=g(g(k))$

$\Rightarrow u''(f(k^*)+(1-\delta)k^*-k^*)\cdot[f'(k^*)+(1-\delta)-g'(k^*)] = \beta u''(f(k^*)+(1-\delta)k^*-k^*)\cdot[f'(k^*)g'(k^*)+(1-\delta)g'(k^*)$
$-[g'(k^*)]^2]\cdot[f'(k^*)+(1-\delta)] + \beta u'(f(k^*)+(1-\delta)k^*)\cdot[f''(k^*)g'(k^*)]$

$= u''(f(k^*)-\delta k^*)\cdot[f'(k)+(1-\delta)] - u''(f(k^*)-\delta k^*)\,g'(k^*) = \beta u''(f(k^*)+(1-\delta)k^*)$
$\cdot(f'(k^*)+(1-\delta))^2\cdot g'(k^*) - \beta u''(f(k^*)-\delta k^*)\cdot(f'(k^*)+(1-\delta))[g'(k^*)]^2 + \beta u'(f(k^*)-\delta k^*)\cdot$
$f''(k^*)[g'(k^*)]$

$= \beta u''(f(k^*)-\delta k^*)(f'(k^*)+(1-\delta))\cdot\underline{[g'(k^*)]^2} - [u''(f(k^*)-\delta k^*)+\beta u''(f(k^*)-\delta k^*)\cdot(f'(k^*)+(1-\delta))^2$
$+\beta u'(f(k^*)-\delta k^*)\cdot f''(k^*)]\ \underline{g'(k^*)} + u''(f(k^*)-\delta k^*)[f'(k^*)+(1-\delta)] = 0$

thus we have the desired quadratic in terms of $g'(k^*)$

assume $u(c)=\dfrac{c^{1-\gamma}}{1-\gamma}$, $u'(c)=-c^{-\gamma}$, $u''(c)=-\gamma c^{-\gamma-1}$

$f(k)=Ak^{\alpha}$ $\quad f'(k)=\alpha Ak^{\alpha-1}$ $\quad f''(k)=(\alpha^2-\alpha)Ak^{\alpha-2}$

$\beta=.96$, $A=1$, $\gamma=.5$ $\quad \alpha=.33$ $\quad \delta=.05$ at this point I plugged those values into Mathematica and got $\boxed{-.2288\ g'(k^*)^2+.4662\ g'(k^*)-.2377=0}$

Solving the above quadratic in terms of $g'(k^*)$ yields

$$\boxed{\begin{array}{l} g'(k^*)=1.01874+.030951\,i \\ \text{and} \\ g'(k^*)=1.01879-.03051\,i \end{array}}$$

that is $g'(k^*)\in\mathbb{C}$

should be real

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