Economics 205 A

Problem Set No. 3

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Problem No. 1

N Person Economy - Identical Preferences

We are considering the growth model for N identical consumers instead of the single consumer that we discussed in class. Assume each consumer is given the same initial emdowment k_0, $\implies k_{i0} = k_0 \ \forall i$. Further, assume $u_i \equiv u \ \forall i$. Let

$$\sum_{i=1}^{N} k_{it} = K^A \quad \text{and} \quad \sum_{i=1}^{N} h_{it} = H^A, \ \forall t.$$

We can define an Arrow-Debreu equilibrium as

a list of sequences

$\{c_{it}\}_{t=0}^{\infty}, \{k_{it}^C\}_{t=0}^{\infty}, \{h_{it}^C\}_{t=0}^{\infty}, \{k_{it}^F\}_{t=0}^{\infty}, \{h_{it}^F\}_{t=0}^{\infty}, \{i_{it}\}_{t=0}^{\infty}, \{y_t\}_{t=0}^{\infty}, \{p_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty},$

such that

1. Consumer Max: Taking prices $\{p_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}$ as given $\{c_{it}\}_{t=0}^{\infty}, \{i_{it}\}_{t=0}^{\infty}, \{k_{it}^C\}_{t=0}^{\infty}, \{h_{it}^C\}_{t=0}^{\infty}$, Solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_{it}) \quad \text{subject to} \quad \sum_{t=0}^{\infty} (p_t c_{it} + p_t i_{it}) = \sum_{t=0}^{\infty} (w_t h_{it}^C + r_t k_{it}^C),$$

with law of motion $k_{i(t+1)}^C = (1-\delta) k_{it}^C + i_{it}, \ c_{it} \geq 0, \ 0 \leq h_{it}^C \leq 1, \ k_{i0}$ given.

2. Firm Max: Taking prices $\{p_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}$ as given $\{y_t\}_{t=0}^{\infty}, \{K_t^A\}_{t=0}^{\infty}, \{H_t^A\}_{t=0}^{\infty}$, solve

$$\max \sum_{t=0}^{\infty} p_t y_t - r_t K_t^A - w_t H_t^A \quad \text{subject to} \quad y_t = F(K^A, H^A),$$

with $K^A \geq 0, H^A \geq 0$.

3. Markets Clear

$$\sum_{i=1}^{N} h_{it}^{C} = h_t^F = H^A, \ \sum_{i=1}^{N} k_{it}^{C} = k_t^F = K^A, \ \sum_{i=1}^{N}(c_{it} + i_{it}) = y_{it} \ \forall t.$$

Now let's solve for the equilibrium of the N agent economy. Let's start with the consumer's problem[1]. Let $h_{it} \equiv 1 \implies H^A = N$, and rearrange the law of motion to $i_{it} = k_{i(t+1)} - (1-\delta)k_{it}$. The Lagrangian for the consumer's problem is

$${}^{C}(c_{it}, k_{it}, \lambda_t) \equiv \sum_{t=0}^{\infty} \beta^t u(c_{it}) + \lambda_t \left[\sum_{t=0}^{\infty} [r_t k_{it} + w_t + p_t(1-\delta)k_{it}] - \sum_{t=0}^{\infty} p_t(c_{it} + k_{i(t+1)}) \right] \ \forall i,$$

whose first order conditions are

$$\frac{\partial \Lambda^C}{\partial c_{it}} \equiv \beta^t u'(c_{it}) = \lambda_t p_t \tag{1}$$

$$\frac{\partial \Lambda^C}{\partial k_{it}} \equiv \lambda_t p_{t-1} = \lambda_t \left[r_t + p_t(1-\delta) \right] \tag{2}$$

$$\frac{\partial \Lambda^C}{\partial \lambda_t} \equiv \sum_{t=0}^{\infty} [r_t k_{it} + w_t + p_t(1-\delta)k_{it}] = \sum_{t=0}^{\infty} p_t(c_{it} + k_{i(t+1)}). \tag{3}$$

The firm faces the problem

$$\max p_t F(K_t^A, H_t^A) - r_t K_t^A - w_t H_t^A \quad \text{subject to} \quad y_t = F(K_t^A, H_t^A),$$

and with the inelastic labor supply implemented, can be written as

$$\max p_t F(K_t^A, N) - r_t K_t^A - w_t N \quad \text{subject to} \quad y_t = F(K_t^A, N),$$

The Lagrangian for this problem is

$$\Lambda^F(K_t^A, N, \lambda_t) \equiv p_t F(K_t^A, N) - r_t K_t^A - w_t N + \lambda_t \left[F(K_t^A, N) - y_t \right],$$

whose first order conditions are

$$\frac{\Lambda^F}{\partial K_t^A} \equiv p_t \frac{\partial F}{\partial K_t^A} = r_t \tag{4}$$

$$\frac{\Lambda^F}{\partial N} \equiv p_t \frac{\partial F}{\partial N} = w_t \tag{5}$$

$$\frac{\Lambda^F}{\partial \lambda_t} \equiv F(K_t^A, N) = y_t \tag{6}$$

From the manipulation of (1), we get

$$\frac{\beta^t u'(c_{it})}{\beta^{t-1} u'(c_{i(t-1)}} = \frac{p_t}{p_{t-1}}.$$

[1] For ease, let $\lambda_{it} \equiv \lambda_t$.

From the manipulation of (2), we get

$$\frac{p_{t-1}}{p_t} = \frac{r_t}{p_t} + (1-\delta).$$

From the manipulation of (4), we get

$$\frac{\partial F}{\partial K_t^A} = \frac{r_t}{p_t},$$

which, when combined with the manipulation of (1), yields

$$\frac{u'(c_{it})}{\beta u'(c_{i(t-1)})} = \frac{\partial F}{\partial K_t^A} + (1-\delta) \implies \frac{\partial F}{\partial K_t^A} = \frac{u'(c_{it})}{\beta u'(c_{i(t-1)})} - (1-\delta),$$

which is equivalent to the same equilibrium solution for the single agent economy discussed in class. Hence, in this equilibrium prices are also $\{p_t^*\}_{t=0}^{\infty}, \{w_t^*\}_{t=0}^{\infty}, \{r_t^*\}_{t=0}^{\infty}$ and each consumer makes choices $\{c_t^*\}_{t=0}^{\infty}, \{k_t^*\}_{t=0}^{\infty}, \{h_t^*\}_{t=0}^{\infty}, \{i_t^*\}_{t=0}^{\infty}$.

Problem No. 3

AD vs. SoM II

Assume preferences

$$\sum_{t=0}^{\infty} \beta^t u_i(c_t)$$

Further, endowments of each agent $(\omega_{i1}, \omega_{i2})$ are given by

$$\omega_i \equiv (\omega_{i1}, \omega_{i2}, ...).$$

An Arrow-Debreu equilibrium for this economy is a list of sequences[2] ... which solve

$$\max_{c_{it}} \sum_{t=0}^{\infty} \beta^t u_i(c_t) \quad \text{subject to} \quad \sum_{t=0}^{\infty} p_t c_{it} = \sum_{t=0}^{\infty} p_t \omega_{it},$$

with market clearing conditions

$$c_{1t} + c_{2t} = \omega_{1t} + \omega_{2t} \ \forall t.$$

A Sequence of Markets equilibrium is a list of sequences[3] ... which solve

$$\max_{c_{it}, b_{it}} \sum_{t=0}^{\infty} \beta^t u_i(c_t)$$

subject to

$$\begin{aligned} p_0 c_{i0} + q_0 b_{i0} &= p_0 \omega_{i0} \\ p_t c_{it} + q_t b_{it} &= p_t \omega_{it} + p_t b_{i(t-1)}, \ t \geq 1, \end{aligned}$$

with market clearing conditions

$$c_{it} = \omega_{it} \quad \text{and} \quad \sum_{t=0}^{\infty} b_{it} = 0.$$

For the Arrow-Debreu equilibrium, we are given a set of endowments for which we can set up a Lagrangian:

$$\Lambda(c_{it}, \lambda) \equiv \beta^t \ln c_{it} + \lambda \left(\sum_t p_t(\omega_{it} - c_{it} \right),$$

whose first order conditions are

$$\frac{\partial \Lambda}{\partial c_{it}} = \frac{\beta^t}{c_{it}} = \lambda p_t \implies \frac{\beta^{t-1}}{c_{i(t-1)}} = \lambda p_{t-1} \tag{7}$$

$$\frac{\partial \Lambda}{\partial \lambda} = p_t c_{it} - p_t \omega_{it} = 0. \tag{8}$$

[2] Abbreviated for ease.

[3] Same as above.

In the interest of the deadline for this problem set[4], the solution is

$$
\begin{aligned}
c_1^* &= \frac{1-\beta}{p_0}\left(\frac{p_0}{1-\beta} + \frac{\beta p_0}{1-\beta^2}\right) = 1 + \frac{\beta}{1+\beta} = \frac{1+2\beta}{1+\beta} \\
c_2^* &= 3 - \frac{1+2\beta}{1+\beta} = \frac{2+\beta}{1+\beta}
\end{aligned}
$$

[4]The pencil work is attached behind this page.

For the Sequence of Markets equilibrium, we get the Lagrangian

$$\Lambda(c_{it}, b_{it}, \lambda_t) \equiv \beta^t \ln c_{it} + \lambda_t[p_t(\omega_{it} + b_{i(t-1)} - p_t c_{it} - q_t b_{it}],$$

whose first order conditions are

$$\frac{\partial \Lambda}{\partial c_{it}} = \frac{\beta^t}{c_{it}} - \lambda_t p_t = 0 \tag{9}$$

$$\frac{\partial \Lambda}{\partial b_{it}} = -\lambda_t q_t + \lambda_{t+1} p_{t+1} = 0 \tag{10}$$

$$\frac{\partial \Lambda}{\partial \lambda_t} = p_t(\omega_{it} + b_{i(t-1)}) - p_t c_{it} - q_t b_{it} = 0 \tag{11}$$

From (9), we get

$$p_t(\omega_{it} + b_{i(t+1)} - c_{it}) = q_t b_{it}.$$

From (10), we get

$$\lambda_{t+1} p_{t+1} = \lambda_t q_t.$$

From $\frac{c_{i(t+1)}}{\beta c_{it}}$, we get

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{p_t}{p_{t+1}} \beta \frac{c_{it}}{c_{i(t+1)}}$$

which implies

$$q_t = \beta p_t$$

in the steady state. We now have

$$\frac{p_t}{p_{t+1}} = \frac{1}{\beta} \implies \beta p_t = p_{t+1} \implies q_t = p_{t+1}$$

Recall from (9), we got

$$\begin{aligned} p_t(\omega_{it} + b_{i(t+1)} - c_{it}) &= q_t b_{it} \\ \implies \omega_{it} + b_{i(t+1)} - c_{it} &= \frac{q_t b_{it}}{p_t} \\ \implies c_{it} = \omega_{it} + b_{i(t+1)} &- \frac{q_t b_{it}}{p_t}, \end{aligned}$$

which when combined with our steady state expression above, implies

$$c_{it} = \omega_{it} + b_{i(t+1)} - \beta b_{it}.$$

In words, the value agent i attaches to borrowing today is offset by discounted value tomorrow. Hence, if we impose

$$\beta b_{it} = b_{i(t+1)},$$

then

$$c_{it} = \omega_{it},$$

as desired.

Problem No. 4

Corner Solution!

The social planner's problem is

$$\max_c \sum_{t=0}^{\infty} \beta^t c_t \quad \text{subject to} \quad k_{t+1} \leq f(k_t) + (1-\delta)k_t - c^*$$

If $\exists$ a steady state solution k^* and c^*, then the social planner maximizes her infinite horizon problem by setting $k^* = f(k_0) + (1-\delta)k_0 - c^*$. Given this, there are two possible scenarios depending on k_0 and k^*.

1. If the steady state capital level is feasible, i.e., if

$$k^* \leq f(k_0) + (1-\delta)k_0,$$

   then the social planner will immediately place $k_1 = k^*$. All leftover capital will immediately be consumed as c_1.

2. If the steady state of capital is greater than the initial production rate minus depreciation, i.e., if k^* is not feasible,

$$k^* \geq f(k_0) + (1-\delta)k_0,$$

   then the social planner obviously cannot place $k_1 = k^*$. Instead, she will optimize by keeping $c^* = 0$ until the capital stock reaches k^*.

The last sentence of the problem asks under what conditions will the social planner set $k_2 = k^*$, if

$$\begin{aligned} k^* = k_2 &= f(k_1) + (1-\delta)k_1 \\ &= f(f(k_0) + (1-\delta)k_0) + (1-\delta)(f(k_0) + (1-\delta)k_0). \end{aligned}$$

Given the above, the dynamics of her capital accumulation will be such that $c_0 = c_1 = 0$. At $t = 2$, $k = k^*$.

Problem No. 5

Stability of the Neoclassical Growth Model

Algebra work is preceded by the attached solution we derived using MatLab.