Econ 205A: Problem Set #6

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The following problems are due in review on Friday, November 13, 2009. Please turn in one assignment per group[1].

1 Growth with Labor Supply

Consider the growth model with preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)$$

where c_t is consumption and h_t is hours worked.

Technology is given by:

$$F(k_t, h_t) = k_t^{\alpha}(A^t h_t)^1 - \alpha$$

$$k_{t+1} = x_t + (1 - \delta)k_t$$

Note that growth in this economy is due to the labor augmenting technical change A^t.

Feasibility implies:

$$c_t + k_{t+1} = F(k_t, A^t h_t) - (1 - \delta)k_t$$

The consumer has a budget constraint given by:

$$\sum_{t=0} p_t(c_t + x_t) = \sum_{t=0}^{\infty} p_t(w_t h_t - r_t k_t)$$

[1]These problems are based on problems written by Fernando Alvarez.

1. Write down and solve the consumer's maximization problem. In particular find the first order conditions with respect to c_t, h_t, and k_t. Simplify to find the Euler equation and the relationship between the marginal rates of substitution between c_t and n_t and the wage w_t.

2. A balanced growth path in this economy is given by an initial capital level k_0 and A such that it is optimal to set:

$$c_t = c_0 A^t$$
$$k_t = k_0 A^t$$
$$n_t = n_0$$
$$w_t = w_0 A^t$$
$$r_t = r_0$$

   Write down the FOC for the household imposing a balanced growth path. Use the FOC for the household and firm's problem and feasibility to write down a system of equations in each variable that a balanced growth path must satisfy. You should have an equation for each variable listed above.

3. Show that if preferences are of the form:

$$u(c, 1-h) = \frac{c^{1-\gamma} - 1}{1-\gamma} v(1-h)$$

   for $\gamma \neq 1$, or

$$u(c, 1-h) = \log c + v(1-h)$$

   then there is a balanced growth path.

2 Lucas-Uzawa Growth Model

Let $N(t)$ be the size of the population at time t and n be the growth rate of the population so that:

$$\dot{N}(t) = nN(t)$$

Let the preferences for the representative "dynasty" be given by:

$$\int_0^\infty N(t) e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt$$

where $c(t)$ is per capita consumption.

Production occurs using a Cobb-Douglas function of capital and efficiency units of labor:

$$F(K(t), h(t)u(t)N(t)) = AK(t)^\alpha [h(t)u(t)N(t)]^{1-\alpha}$$

In this formulation $K(t)$ is the aggregate stock of physical capital, $h(t)$ is the per-person stock of human capital, $u(t)$ is the fraction of time devoted to production (so that $1-u(t)$ is devoted to the production of human capital).

The economy's resource constraint is given by:

$$\dot{K}(t) + N(t)c(t) = AK(t)^{\alpha}[h(t)u(t)N(t)]^{1-\alpha}$$

In this formulation we assume that physical capital does not depreciate. Therefore, $\dot{K}$ is investment and hN is the total number of efficiency units of time available for production.

Human capital accumulates according to:

$$\dot{h}(t) = \delta h(t)[1-u(t)]$$

where $1-u(t)$ is the fraction of time devoted to learning and δ is the depreciation rate on human capital.

1. Write down the Hamiltonian for this problem.
2. Take first first order conditions and write down the law of motion for the co-states (note: there are two co-states and two controls).
3. The numbers N, K, h, c, u, and κ are a balanced growth path if:

$$K(t) = e^{(n+\kappa)t)K}$$

$$h(t) = e^{\kappa t}h$$

$$c(t) = e^{\kappa t}c$$

$$u(t) = u$$

   is optimal for initial conditions $K(0) = K$, $h(0) = h$, and $N(0) = N$.

   Write down a formula for the growth rate κ, the time devoted to production u, and the savings rate $\dot{K}/(AK^{\alpha}[huN]^{1-\alpha})$ in a balanced growth path. Solutions should be functions of the parameters: ρ, n, δ, γ, and α.
4. How does κ change as a function of ρ, δ, γ, and n? How does u change as a function of α? Give an intuitive explanation for each case.
5. Write down the relevant transversality conditions. What conditions are needed on the parameters ρ, n, δ, γ, and α so that the balanced growth path characterized in the last two parts satisfies these conditions?