# October 9th 2009 – Lecture Four

Steady State of Arrow-Debreu model. SS in CE. Growth Model dynamics. Finding the steady state. Continuous Time introduced. State (stock) & Control (flow) variables. Step-by-step recipe the find Necessary & Sufficient Conditions.

## Steady State of Arrow-Debreu Model

A wrong definition of SS is an SS Equilibrium such that $c_t, k_t, i_t, h_t, y_t, w_t, p_t, r_t$ are all constant.
What's wrong with that? Prices should not be constant.

**A Steady State Competitive Equilibrium**
- is a value $k_0$
- And an equilibrium $\{c_t, k_t, i_t, h_t, y_t, w_t, p_t, r_t\}$
$such\ that\ k_t = k_0 \quad \forall t$

Think about capital as a state variable (one of a set of variables that describes the state of a dynamic system. Given these coordinates and their FOCs, you can usually determine where the system is headed)

CE is where your state variable(s) are constant. In the Neoclassical Model—fortunate for 1st-year phd students who have trouble with maths—the only state variable is capital. Other (more useful models) have more state variables. But in them, CE is where they are all constant. But *prices* are not states variable, they might not be constant.

**Growth Model Dynamics**
(see graph in notes)

$k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$ ~A law of motion for what capital looks like next period
For any value of k0 you can read off what the transition dynamics are going to be, where capital and consumption are headed in the next period, the next & the next toward our steady state.

*Where are our steady states?*
- (0,0). But that's not interesting.
- And where $k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$

*Dynamics* questions we're interested in…
– is the steady state stable? If you state at the steady state and perturb yourself off it slightly, do you go back? Or explode away. Stable SSs converge back.

*Steady States* – we know there are two. One stable and one unstable.
(0,0) is unstable
$(k^*, c^*)\ needs\ to\ be\ stable$

Let $k_{t+1} = g(k_t)$ - think for some value of $k_t$, this is the optimal value for $c_t$

What is the condition on $g(k_t)$ to ensure stability? To answer this, think about the types of slopes that ensure the dynamics converge toward our $k^*$? Where k crosses $k_{t+1} = g(k_t) = f(k_t) - c_t + (1-\delta)k_t$

Answer: $0 < g'(k^*) < 1$ - you want the steady state to converge back (an really, g' could be negative, but then you might get in to weird limit cycle dynamics. Spirals back into the SS)

**Finding Our Steady States** - FOCs back in the Euler's Equation

■$u'(f(k_{t-1}) + (1-\delta)k_{t-1} - k_t) = \beta u'(f(k_t) + (1-\delta)k_t - k_{t+1})[f'(k_t) + (1-\delta)]$

Now setting $k_{t-1} = k$ ; $k_t = g(k)$ ; $k_{t+1} = g(g(k))$ …

$u'(f(k) + (1-\delta)k - g(k)) = \beta u'(f(k_t) + (1-\delta)g(k) - g(g(k)))[f'(g(k)) + (1-\delta)]$

What do we need to do to this to show that the solution to this is stable?

One (0,0), and another solution that we've done the Lagranian to get this whole time.

The Steps to show the solution is:

1) You need to totally differentiate this function with respect to k

There you'll get a bunch of terms with g'(k). If you collect the terms what comes out is a quadratic function with arguments that are g'(k) (if you let g'(k) = x then you get the quadratic function in x… )

2) There are two solutions. Our steady states. One at (0,0). And another solution that we've done the Lagranian to get this whole time.
3) You'll then need to show that one of the SS (the one we're interested in) has the properties that $g'(k^*) < 1$

## Continuous Time

### State Control Set-Up

$u$ – **Stocks – State Variables –** example, capital

$x$ – **Flows** – **Control Variables** –These are consumption, output – an amount they choice each period

Let $h$ be the return function that depends on the vector of states, $x$

$x \in X \subset \mathbb{R}^m$

And a control vector

$u \in U \subset \mathbb{R}^n$

The problem is

$$\max_{u(t)} \int_0^\infty e^{-\rho t} h(x(t), u(t)) dt$$
$$s.t. \dot{x}(t) = g(t), u(t)$$
$$u(t) < u$$
$$for\ t \geq 0, x_0\ is\ given$$

$\rho$ is a discount factor like $\beta$.
Where $\beta = (is\ something\ like) \frac{1}{1+r}$,
(where r is the interest rate)
$\rho$ $is\ a\ continuous\ time\ discount$

the derivative of x with respect to time. You're going to have a law of motion based on what the state is, and the choices you've made.

$u(t) < u$ – this just ensures that our choices are feasible.

**Note – on exams and in homework**, we'll only be asked to follow this recipe. If you're interested how you go from the discrete time Lagrangian set-up to this set-up, Barro has a great appendix that walks you through setting up limits in those Lagrangian. If you're interested in the mathematical set-up go there.

**Recipe to Find Necessary and Sufficient Conditions –**to solve the problem
First off, assume that all these are functions of time

1) Defining $\lambda(t)$ as the vector of co-state variables,
   write down the current-value Hamiltonian as:
   - $H(x,u,\lambda) = h(x,u) + \lambda\, g(x,u)$

Interpretation: $e^{-\rho t}\lambda(t)$ is the marginal value at time 0, of an increase in x (in the state) at time t.
These are all shadow values of the states. Depending on how the problem is formulated, the shadow value could be the M.V. of t of an increase in x (or you could do it all in time zero prices).

2) Now we're going to look for FOCs. In maximization with respect to the control, take the Hamiltonian of with respect to u it should equal zero. This is the law of motion for the state variable.
   $H_u = h_u + \lambda g_u = 0$

3) The condition on the co-state. Write down Law of Motion for the co-state
   - $\dot{\lambda} = \rho\, \lambda(t) - H_x(x,u,\lambda)$
     $= \rho\, \lambda - h_x - \lambda\, g_x$

4) Transversality Condition given by:
   $\lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0$

FYI – the discrete time version of the transversality condition (the 2 are saying sort of the same time)
$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$
A good write up on this is in appendix of Barro book

■Main thing to take away from TVC
is that we can't have explosive, forever increase solutions.

To sum up these conditions, we need the first order condition (1), the law of motion for the state (2)(this is the initial optimality condition), the law of motion for the co-state (3), and then the transversality condition is a regularity assumption.
These conditions are necessary (with regularity assumptions) and sufficient (given the problem is convex) to find the path of x and u to be optimal.