November 16th 2009 - Lecture Sixtee006E

Econ205A Macro - 20091116

Outline – Review continued of Problem Set 6 Part 2. Mathematical conditions needed for dynamic programming to work.

Problem Set Six Review

Having gone over all our problem sets Gorry gives us intuition into our problem set models.

Example with Mathematical Conditions Necessary for Dynamic Programming to Work

In the last lecture we did two examples of setting up Bellman equations. We'll get more of that in Problem Set Seven. Today we'll continue with another example and consider the mathematical conditions that are required for dynamic problem to work.

Consider the following max problem – individual consumer maximizing their stream of consumption, an asset accumulation problem

$$\max_{c_t, x_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.t. \quad x_{t+1} = w_t - c_t + (1+r_t)x_t$$
$$\& \quad x_{t+1} \geq -D$$

$x_t: assets$
$w_t: wage\ income,$ $a\ given\ sequence, we\ aren't solving\ for\ this$
$r_t\ interest\ rate$
$$\beta = \frac{1}{1+\rho} \in (0,1)$$
D is just some maximum debt limit. (we are putting a cap on the amount of debt one can accumulate.

We're going to use this simple single agent maximization problem over how much to consume each period knowing their wages and how assets evolve over time. A basic savings/consumption tradeoff optimization problem. Wages & interest rates can change over time.

In last class we wrote out all the steps to solve a dynamic programming problem like this. Let's start with step one.

1) Write out the sequence problem

You can define the problem as the value given some initial assets x_0.

$$V(x_0) \equiv \max_{x_{t+1}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$s.t. \quad x_{t+1} \in \Gamma(x_t)$$

First, you can substitute out consumption and plug it in to the utility function. You have something that involves $x_t\ \&\ x_{t+1}$ – your assets today and your assets tomorrow. We also need to acknowledge that x_{t+1} is in the feasible set Γ which depends on your state today (x_t).

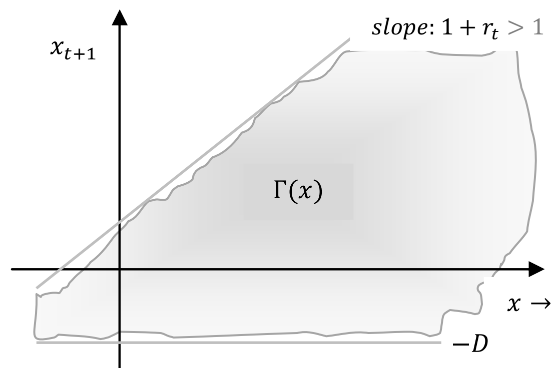
Rewriting,

$$V(x_0) = \max_{x_{t+1}} \sum_{t=0}^{\infty} \beta^t F(w_t + (1+r_t)x_t - x_{t+1})$$

$$\Gamma(x_t) = \begin{matrix} \{x_{t+1} : -D \leq x_{t+1} \leq w_t + (1+r_t)x_t\} \\ [m, M(x_t)] \end{matrix}$$

The Feasible Set Γ — what is x_{t+1} able to be? At the low end it is the max debt level $-D$ and at the high end it is if the agent consumed nothing, $w_t + (1+r_t)x_t$.

Meaning your choice at x_t is bounded.



The graph shows the choice set that is available. $\Gamma(x_t)$ bounded below by the debt limit and above by max growth rate $w_t + (1+r_t)x_t$. x_{t+1} $intercept: w_t$

2) Choose State Variables, Write the Bellman Equation.

The state variable is

$x \in X, \qquad x \in [\bar{x}, \infty)$

Where does x reside? Here there is a lower bound, but it can go to infinity.

Bellman Equation

$$V(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta V(y)$$

Your utility function today plus the discounted value of going into the next period with assets y.
This gives us one static problem to solve. The challenge is that we're solving for V, a functional equation. The Solution to the problem is another equation.

$$The\ function\ V\ is\ unknown$$

The question you want to know?

Is there a solution?
Is it unique?

We want properties that we can check to ensure we get these two things. Usually if the conditions hold that there is a solution, the solution is almost always unique.

3) What sort of condition do we need?

Conditions, such that,

$$V^*\ is\ the\ unique\ solution$$

The rest of the lecture goes over the conditions in an example sense, skipping the heavy math that proves the conditions work. You can find that sort of thing interesting, Lucas/Stokey and other textbooks are a good place to turn to.

We'll be going over the conditions that you'll need to check.

Condition for a Unique Solution

We need solutions that our choice set is bounded – then we can apply the Weierstrass Theorem.

In general with existence proofs, you want to bound the constraint set so that you have a bounded set of choices and if you are looking for a maximum or a minimum over those set of choices, you would at least know that on a bounded set those things exist.

$$\begin{aligned}&for\ each\ x \in X\ (for\ each\ state\ you\ can\ be\ in\ in\ the\ state\ space, we\ need\ a\ condition\ on\ Gamma)\\ &\Gamma(x) = [m(x), M(x)]\\ &where\ m(x) \leq M(x)\ \&\ both\ are\ finite\end{aligned}$$

One condition is that for every x –for every state that you choose—you have lower bound and an upper bound in your choice set, hence they both need to be finite.

- $m\ (alternately, x \subset \mathbb{R}^l\ convex\ or\ X\ is\ a\ coutable\ set)$

$\mathbb{R}^l$ means that x is in some vector space l. l is like, if x is l dimensional state vector.

Alternative assumptions: $x \subset \mathbb{R}^l\ convex\quad - or -\quad X\ is\ a\ coutable\ set$
For detailed readings and proofs, turn to Stokey, Lucas & Prescott.

Now, the first two are satisfied in the example we have above.

- $m(x),\ \ M(x)\ \ are\ continuous\ functions$
- $F(x,y)\ is\ continuous\ on\ its\ domain$
- $0 < \beta < 1$

This condition means that the future must be less important. Mathematically this condition is going to be necessary to have a contraction – so the same thing that gives you discounting is also important for the solution methods that are used – you have to discount future consumption.

- $\beta\ \ must\ dominate\ growth\ rate\ of\ economy, \left(\kappa = \frac{\dot{c}}{c}\right)$

If you're in a growing economy, the discounting must dominate the growth rate of the economy. That's required to have a contraction. Beta must be enough less than one that the growth rate doesn't make things bigger over time – the growth rate will be something like $1 + r_t$ & beta will be something like $\frac{1}{1-\rho}$.

If you have all these conditions then the contraction mapping theorem applies,

$$\begin{aligned}&(Tf)(x) = \max_{y\in\Gamma(x)} F(x,y) + f(y)\\ &where\ T\ is\ a\ contration\ has\ a\ unique\ fixed\ point\end{aligned}$$

Then we know if V^* is a solution, we have that it's a unique solution based on this theorem
The contraction mapping theorem under these conditions is going to give us a fixed point, which is a solution, to the value function. It's also going to give us a unique solution.

The contraction mapping theorem is what's giving us an algorithm to solve these types of problems. Meaning,

$$\textit{we can calculate } V^* \textit{ with successive approximations}$$
$$\textit{define the sequence } V_n \textit{ by:}$$
$$V_{n+1} = \max_{y\in\Gamma(x)} F(x,y) + \beta V_n(y)$$

Given any function V_0 , we can compute a sequence of functions. We can guess something (say V_0 is a constant, or V_0 is increasing in y ...) then this problem easier to solve. We just guess some function, take the derivative and solve for the optimal y at every point. That computes a new function by plugging in the optimal y at each point.

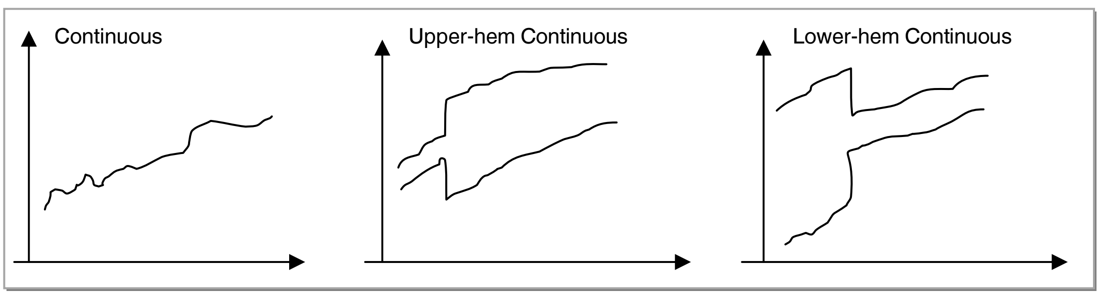
We compute this sequence of functions and the result is,

$$\{V_n\} \to V^* \textit{ and it converges at rate } \beta$$

The sequence of functions, V_n, converges to V^*
Moreover we know the rate of convergence is at rate β.

Summing up what we have so far

$$\textit{We have properties such that } (s.t.)$$
$$\textit{the set of maximizers } G(x) \textit{ is non} - \textit{empty, closed, bounded, upper} - \textit{hem: continuous}$$



4) We want assumptions on F, Γ so that there are nice properties on our optimal solution.
V should be monotone, concave (gives a unique maximize), and differentiable. Going one by one,

a) **Monotonicity** – Think about contraction on a value function. If you want the value function to be increasing in x, what conditions make sense for that to hold?

Firstly you'll need your objective function to be increasing in x – if you're better off with more of the state-variable then your value will be higher in the state variable.

What can go wrong for this not to hold? We showed that $\Gamma(x)$, the choice set, gets larger with more x. So it could be that you like more x but having more x gives you less choices, making you worse off. So two parts to the objective function – how much utility you get and now that affects your choices for the future.

i) For monotonicity, suppose $F(\cdot, y)$ is strictly increasing in x.
ii) If $x \le x'$, $\textit{then } \Gamma(x) \subset \Gamma(x')$, then V is strictly increasing.

b) **Concavity**: For $x, x' \in X$, $y, y'\ \Theta \in (0,1)$
$x_\Theta = \Theta x + (1-\Theta)x'$, $y_\Theta = \Theta y + (1-\Theta)y'$

So x_Θ & y_Θ are just the convex combinations x & x' *and* y & y' respectively.

Suppose

i) Γ *is convex in the sense that if* $y \in \Gamma(x)$ *and* $y' \in \Gamma(x')$ *then* $y_\Theta \in \Gamma(x_\Theta)$ *ie, the graph of* Γ *is convex*

ii) F is concave in the sense that if $\mathrm{y} \in \Gamma(\mathrm{x})$ and $\mathrm{y}' \in \Gamma(\mathrm{x})$, then $\mathrm{F}(\mathrm{x}_\Theta, \mathrm{y}_\Theta) \geq \Theta \mathrm{F}(\mathrm{x}, \mathrm{y}) + (1 - \Theta)\mathrm{F}(\mathrm{x}, \mathrm{y})$
With strict inequality if $x \neq x'$
then V *is strictly concatve* $-$ $g(x)$ *is an optimal policy function*

Why is this important?

Concavity ensures that g is an optimal policy function — that it's single valued.

Secondly, if you do the value function iteration to find a solution, you can define the optimizer, say $g_n(x)$, as the optimal policy for each iteration of that step and that $g_n's$ will also converge. So our optimal policies is going to converge.

We'll start up with differentiability on Wednesday…