Economics 205 A

Problem Set No. 7

Maggie apRoberts-Warren
Carmen Hernandez-Lara
Kat Lopiccalo
Jesse Bigley
Curtis Kephart
A. Christopher Limnios

Aspen Gorry, Instructor

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Problem I

Durable Good Consumption

This question asks you to write out the Bellman equation for each specification of the growth model that contains a durable good. In each case, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, d_t)$$

and the individual is endowed with one unit of time that is supplied as labor in each period. For each specification of technology, write down the Bellman equation that could be used to solve the problem (you do not have to do the analysis).

1. *Putty-putty: Output can be used as durables, capital and consumption. Moreover, capital and durables can be turned back into output in any period. Therefore, feasibility requires:*

   $$c_t + d_{t+1} + k_{t+1} \leq F(k_t, h_t) + (1 - \delta^d)d_t + (1 - \delta^k)k_t,$$

   $$k_t, d_t, c_t \geq 0.$$

   The sequence problem describing this economy is

   $$\max_{c_t, d_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, d_t)$$

   subject to

   $$c_t + d_{t+1} + k_{t+1} \leq f(k_t) + (1 - \delta^d)d_t + (1 - \delta^k)k_t$$

$$k_t, d_t, c_t \geq 0.$$

The state variables for this economy are k_t and d_t and the control variable is c_t. The Bellman equation is

$$\mathbb{V}(d,k) = \max_{c,d',k'} u(c,d) + \beta V(k',d') \quad \text{subject to} \quad d' + k' \leq f(k) + (1-\delta^d)d + (1-\delta^k)k - c$$

$$c, d, k \geq 0.$$

2. *Putty-clay: Output can be used as durables, capital and consumption, however, once in place, durables and capital can no longer be turned into output. Feasibility now requires:*

$$c_t + i_t^d + i_t^k \leq f(k_t, h_t)$$
$$k_{t+1} = (1-\delta^k)k_t + i_t^k$$
$$d_{t+1} = (1-\delta^d)d_t + i_t^d$$

$$c_t, i_t^d, i_t^k \geq 0.$$

The sequence problem describing this economy is

$$\max_{c_t, d_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, d_t)$$

subject to

$$c_t + i_t^d + i_t^k \leq f(k_t, h_t)$$
$$k_{t+1} = (1-\delta^k)k_t + i_t^k$$
$$d_{t+1} = (1-\delta^d)d_t + i_t^d$$

$$c_t, i_t^d, i_t^k \geq 0.$$

The state variables for this economy are k_t and d_t and the control variables are c_t, i_t^d and i_t^k. The Bellman equation is

$$\mathbb{V}(d,k) = \max_{c,d',k',i^{d\prime},i^{k\prime}} u(c,d) + \beta V(k',d')$$

subject to

$$0 \leq c + i^d + i^k \leq f(k)$$
$$k' = (1-\delta^k)k + i^k$$
$$d' = (1-\delta^d)d + i^d$$

$$c, i^d, i^k, d, k \geq 0.$$

3. *Clay-clay: Now each good is produced with its own production function and capital and durables cannot be turned into output. However, capital can be reallocated between the three uses in each period. Feasibility requires:*

$$
\begin{aligned}
c_t &= f^c(k_{ct}, h_{ct}) \\
i_t^d &= f^d(k_{dt}, h_{dt}) \\
i_t^k &= f^k(k_{kt}, h_{kt}) \\
d_{t+1} &= (1-\delta^d)d_t + i_t^d \\
k_{t+1} &= (1-\delta^k)k_t + i_t^k \\
h_{ct} + h_{dt} + h_{kt} &= 1 \\
k_{ct} + k_{dt} + k_{kt} &= k_t \\
c_t, i_t^d, i_t^k &\geq 0.
\end{aligned}
$$

The sequence problem describing this economy is

$$\max_{c_t, k_{t+1}, d_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, d_t)$$

subject to

$$
\begin{aligned}
c_t + i_t^d + i_t^k &= f^c(k_{ct}, h_{ct}) + f^d(k_{dt}, h_{dt}) + f^k(k_{kt}, h_{kt}) \\
d_{t+1} &= (1-\delta^d)d_t + i_t^d \\
k_{t+1} &= (1-\delta^k)k_t + i_t^k \\
h_{ct} + h_{dt} + h_{kt} &= 1 \\
k_{ct} + k_{dt} + k_{kt} &= k_t \\
c_t, i_t^d, i_t^k &\geq 0.
\end{aligned}
$$

The state variables for this economy are k_t and d_t and the control variable is c_t, h_{ct} and h_{dt}. The Bellman equation is

$$\mathbb{V}(d,k) = \max_{c, d', k', h_c', h_d'} u(c,d) + \beta V(k', d')$$

subject to

$$
\begin{aligned}
0 \leq c &\leq f^c(k_c, h_c) \\
k' =& (1-\delta^k)k + f^k(k_k, h_k) \\
d' =& (1-\delta^d)d + f^d(k_t, h_d) \\
h_k &= 1 - h_c - h_d \\
k_c + k_d + k_k &= k.
\end{aligned}
$$

Problem II

Fixed Capital Investment

Two is very similar to part (c) of problem number 1. We differentiate this problem from clay-clay model by establishing that capital cannot move between different types of capital. That is, in the clay-clay model,

$$k_{ct} + k_{dt} + k_{kt} = k_t,$$

meaning you can have capital in one form one period[1], the next period you are free to transform the capital into another form[2]. Thus, we will remove the constraint $k_{ct} + k_{dt} + k_{kt} = k_t$, leaving

$$\begin{aligned} c_t &= f^c(k_{ct}, h_{ct}) \\ i_t^d &= f^d(k_{dt}, h_{dt}) \\ d_{t+1} &= (1-\delta^d)d_t + i_t^d. \end{aligned}$$

Now that the three types of capital are not freely interchangeable, we need three separate laws of motion for k_{ct}, k_{dt} and k_{kt}. Also, when previously capital had one investment source, i_t^k, we must now define three new investment sources associated with our three new laws of motion k_{kt+1}, k_{ct+1} and k_{dt+1}:

$$\begin{aligned} & i_t^{kd} + i_t^k + i_t^c = f^k(k_{kt}, h_{kt}) \\ & k_{dt+1} = (1-\delta^{dk})k_{dt} + i_t^{dk} \\ & k_{kt+1} = (1-\delta^k)k_{kt} + i_t^k \\ & k_{ct+1} = (1-\delta^c)k_{ct} + i_t^c \\ & h_{ct} + h_{kt} + h_{dt} = h_t = 1 \\ & c_t, d_t, i_t^d, i_t^{dk}, i_t^c, i_t^k, k_{dt}, k_{kt}, k_{ct} \geq 0 \end{aligned}$$

The control variables are c_t, i_t^{kd}, i_t^d, i_t^K, i_t^c h_{ct} and h_{dt} and the state variables are k_{dt}, k_{kt} and k_{ct} and d_t. The Bellman equation is

$$\mathbb{V}(d, k_c, k_k, k_d) = \max_{c, i^{kd}, i^d, i^k, i^c, h_c, h_d} u(c,d) + \beta V(k_d', k_c', k_k', d')$$

[1] Let's say you have 100% of your capital in the form of k_{kt}.

[2] Say, now at the next period 100% of capital is in the form of k_{ct}.

subject to

$$
\begin{aligned}
&i^{kd} + i^k + i^c = f^k(k_k, 1 - h_c - h_d) \\
&0 \leq c+ \leq f(k_c, h_c) \\
&d' = (1 - \delta^d)d + f^d(k_d, h_d) \\
&k_d' = (1 - \delta^{dk})k_d + i^{dk} \\
&k_k' = (1 - \delta^k)k_k + i^k \\
&k_c' = (1 - \delta^c)k_c + i^c \\
&h_c + h_k + h_d = 1 \\
&c, d, i^d, i^{dk}, i^c, i^k, k_d, k_k, k_c \geq 0.
\end{aligned}
$$

Problem III

Time to Build

In the growth model discussed in class investments in one period become productive in the next period. This implies that the time to build is one period. Now we will consider a two period time to build model. Assume that in order to increase the capital stock in period $t+2$ by one unit it requires an initial investment of $\frac{1}{2}$ in period t and an initial $\frac{1}{2}$ unit in period $t+1$. In particular, let i_{1t} denote investment in new projects in period t, and let i_{2t} denote investment in ongoing projects in period t. Let s_t denote the stock of unfinished projects in period t that were begun in period $t-1$ ($s_t = 2i_{1t-1}$). Then the law of motion for the (productive) stock of capital is:

$$k_{t+1} = (1 - \delta)k_t + \min\{s_t, 2i_{2t}\}.$$

Preferences are standard:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

and the production function is

$$F(k_t, h_t).$$

1. *Write down a Bellman equation for the social planner's problem.*

   The sequence problem describing this economy is

$$\max_{c_t, k_{t+1}, s_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   subject to

$$
\begin{aligned}
0 \leq c_t &\leq f(k_t) - \min\{s_t, 2_{i2t}\} \\
k_{t+1} &= (1 - \delta)k_t + \min\{s_t, 2_{i2t}\}.
\end{aligned}
$$

The control variables are c_t, k_{t+1} and s_{t+1} and the state variables are k_t and s_t. The Bellman equation is

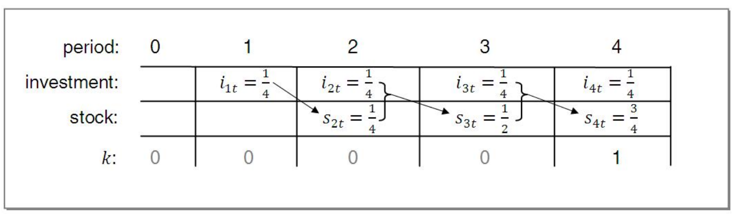
$$\mathbb{V}(k,s) = \max_{c,k',s'} u(c) + \beta V(k',s')$$

subject to

$$\begin{aligned} 0 \leq c &\leq f(k) - i \\ (1-\delta)k \leq k' &\leq (1-\delta)k + \min\{s, 2i_2\} \\ s' &= 2i_1 \\ i_1 + i_2 &= i. \end{aligned}$$

2. *Suppose we generalize to four periods of time to build with $\frac{1}{4}$ unit of investment in each of four periods to produce an additional unit of capital in the fifth period. Formulate the Bellman equation in this case.*

The investment stock flow required for one unit of capital at time 4 has been broken up as follows:



The Bellman is

$$\mathbb{V}(k, s_4, s_3, s_2) = \max_{c,k',s'_4,s'_3,s'_2,i_1,i_2,i_3,i_4} u(c) + \beta V(k, s'_4, s'_3, s'_2)$$

subject to

$$(1-\delta)k \leq k' \leq (1-\delta)k + \min\left\{\frac{4}{3}s_4, 4i_4\right\}$$

$$\begin{aligned} &0 \leq c \leq f(k) - i \\ &i = i_1 + i_2 + i_3 + i_4 \\ &s'_4 = \min\left\{\frac{3}{2}s_3, 3i_3\right\} \\ &s'_3 = \min\{2s_2, 2i_2\} \\ &s'_2 = i_1 \end{aligned}$$